

The Atiyah–Bott–Shapiro Orientation in Bordism and Field Theory

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Abstract

The Atiyah–Bott–Shapiro orientation is a bridge between Spin bordism and K-theory, providing a stable K-theoretic orientation of Spin vector bundles. There are also versions of the orientation for Spin and K-theory over the complex numbers and quaternions. In this thesis, we develop these orientations using Clifford algebras and their modules.

This orientation has a multitude of applications. In stable homotopy theory, this orientation is the foundation for the Anderson–Brown–Peterson splitting. These splittings are maps from the Thom spectra representing Spin bordism (and its variants) to a wedge sum of Eilenberg–MacLane spectra and connective covers of the spectra representing K-theory. After localizing at the prime 2, these maps become equivalences, allowing us to determine the Spin bordism groups. We sketch the construction and proof of these splittings.

Spin bordism (and its variants) also finds applications in physics, where it is closely related to certain types of topological quantum field theories. We will briefly introduce topological quantum field theories and their symmetry types. Then, we proceed by discussing invertible and extended topological quantum field theories, motivating the relationship between these types of theories and Spin bordism.

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Chapter 1

Introduction

Symmetry is a transformation of an object that leaves certain properties unchanged, and it is a rich subject at the heart of mathematics and physics. Common examples of symmetries are translations, rotations, and reflections in the plane. These are familiar to us, as they are quite common in art, architecture, and everyday objects (see Figure 1.1). But symmetry often manifests in more subtle, abstract, and fundamental ways, with profound consequences.

In physics, special relativity states that the laws of physics must be unchanged under certain symmetry transformations of our coordinate system for space and time. This is a powerful tool—using this strong requirement as well as unitarity, a basic property that all quantum mechanical systems must satisfy, Wigner was able to completely classify the different species of particles that could exist [BW48] in a quantum universe compatible with special relativity. Symmetry, or its absence, also has a strong effect on the properties of matter. For example, solids, like crystals and metals, tend to have many discrete symmetries like translations and reflections, making them useful for applications like semiconductors, while liquids and gases tend to be very asymmetric.

In mathematics, symmetry is just as omnipresent, although typically in a more abstract form. Instead of limiting ourselves to symmetries of concrete geometric objects, one is often led to consider symmetries of other mathematical objects like higher-dimensional spaces, algebraic structures, and complex systems. The abstract notion of symmetry is captured by algebraic objects known as **groups**, which encode the result of performing one symmetry transformation after another. The idea of a group can be enriched further by considering its interaction with **topology**, which allows us to see how symmetries can be applied **continuously** or **smoothly**. By studying how mathematical objects interact with groups, we can learn more about both groups and the objects in consideration.

Rotations are a universal form of symmetry and aspect of geometry that we all grapple with in our day to day lives, whether we think about it consciously or not. Whenever we rotate an object with our hands, make a turn while driving a car, or roll a ball, we are using our intuitive understanding of rotations. However familiar and intuitive rotations may seem,



(a) The sides of this building exhibit a translational symmetry.



(b) This bottle cap has rotational and reflectional symmetries.

Figure 1.1: Examples of everyday symmetries.

there are aspects of the nature of rotations that are both truly surprising and have great consequences.

The fact that we will be concerned with is that there are essentially two distinct ways to *make* any rotation. You can see this by holding an object upright in your hand — for example, a glass of water. If you rotate it 360 degrees while keeping it upright by passing it under your elbow, you can make a full rotation of the glass. However, your arm will now be twisted with your elbow pointing up. You can do another full rotation of the glass by rotating your elbow, and now your arm will be in its original state. So, we see that there are essentially two different “ways” of doing a full rotation of an object. This process is depicted in Figure 1.2.

To make sense of this mathematically, we can form the **Lie group** of rotations, which keeps track of how rotations can be performed one after the other as well as what it means to rotate smoothly. Then we can talk about the **fundamental group** of the group of rotations, which is an abstract group that measures the different ways one can gradually do a full rotation. By describing this fundamental group, we will see that this captures the phenomenon of these two distinct ways to do a full rotation.

We can continue even further by introducing **Spin groups**. These are also examples of Lie groups, and they are closely related to groups of rotations, but in some sense they keep track of the way one makes a rotation. These groups are of great importance in topology and



Figure 1.2: One full rotation of the glass returns it to its original state, but leaves the arm in a different configuration. Doing another full rotation of the glass returns both the glass and the arm to their original states.

physics. In topology, we can introduce the notions of **Spin bundles** and **Spin bordism**, which allow us to study the different ways one can form shapes and spaces with Spin data. In physics, Spin groups allow us to describe the quantum mechanical behavior of electrons when we perform rotations, explaining important phenomena like the Pauli exclusion principle. We can also consider physical theories with “internal symmetries” described by Spin groups, which relate to so-called **topological phases**. This topic is intimately connected with Spin bordism.

In this thesis, we will begin by introducing topological field theories, including those of the invertible and extended varieties, and those with a symmetry type described by the Spin groups, touching on the results of [FH21]. After this informal discussion of topological field theories, we will go over the construction of the Spin groups from Clifford algebras, as well as related groups known as the complex and quaternionic Spin groups. We will then continue to discuss the Atiyah–Bott–Shapiro orientation, introduced in [ABS64], which is a crucial relationship between topological K-theory and Spin bordism. We will use this as a starting point to develop the theory of Spin bordism, summarizing the results of [ABP67], which ties into the computational aspects of the topological field theory discussion.

Our approach will be to discuss these topics in as much generality as possible with respect to the algebra we are working over (real, complex, or quaternionic), only working with individual cases when it is necessary. We do this so that we can highlight and exploit the similarities and relationships between these different variants, and so we can give a more streamlined and conceptual exposition of this subject.

1.1 Outline

We begin with an informal discussion of topological quantum field theories. This will start with the introduction of Wick rotation and the definition of a Wick-rotated topological field theory as a symmetric monoidal functor from a bordism category. Then we will explain the notion of topological field theories involving tangential structures on manifolds. Finally, we will define invertible and extended theories, and following [FH21], highlight the importance of Thom (Madsen–Tillman) spectra in the theory and use the results of the calculations from the Anderson–Brown–Peterson splitting to classify deformation classes of field theories with tangential structure defined by the Spin groups (and their variants).

After discussing the applications to physics as motivation, we will then dive into the mathematics that allows us to understand Spin bordism and the calculations we used. To understand where Spin groups come from and their broader context, we introduce Clifford algebras, their modules, and the Spin groups that sit inside them. Our treatment will include a systematic discussion of the isomorphism type of these algebras and a classification of the modules they can support, as well as their relationships under tensor product.

Next, we will define the Atiyah–Bott–Shapiro orientation. This is a connection between

many subjects: K-theory, bordism/Thom spectra, and Clifford algebras, so we summarize the facts of these topics that we need, including the difference bundle in K-theory, which is the main tool used by [ABS64] to build the requisite K-theory classes. Then we will use these tools to construct the Atiyah–Bott–Shapiro orientation and discuss its properties, with an emphasis on its stability and spectrum form.

Lastly, we will use the Atiyah–Bott–Shapiro orientation to construct the Anderson–Brown–Peterson splitting, an important result that allows for a complete determination of the Spin bordism groups. This splitting requires some intensive calculations involving the Steenrod algebra, so we will merely summarize the main points of the construction and provide references to where the long calculations are performed.

We will assume and make use of basic definitions and results from Lie theory and homotopy theory, including stable homotopy theory and characteristic classes, throughout. Our use of the stable homotopy category will be mostly formal, in that the specific model for stable homotopy theory we use does not appear, although we will need a model to discuss the definition of Thom spectra and the maps implementing the Atiyah–Bott–Shapiro orientation. For this, we will use the stable homotopy category of Adams [Ada74] due to its simplicity for our purpose.

We will make use of a small amount of nonstandard notation throughout in order to treat the real, complex, and quaternionic cases on the most even footing possible. Here is a table giving the correspondence between our notation and the standard notation:

$\text{Spin}_{\mathbb{R}}(n)$	$\text{Spin}(n)$	$K\mathbb{R}$	KO	O(1)	$N_{\mathbb{R}}$
$\text{Spin}_{\mathbb{C}}(n)$	$\text{Spin}^c(n)$	$K\mathbb{C}$	K (KU)	U(1)	$N_{\mathbb{C}}$
$\text{Spin}_{\mathbb{H}}(n)$	$\text{Spin}^h(n)$	$K\mathbb{H}$	KSp	Sp(1)	$N_{\mathbb{H}}$

Chapter 2

Spin Bordism and Topological Quantum Field Theories

To motivate a detailed discussion of Clifford algebras, Spin groups, the Atiyah–Bott–Shapiro orientation, and Spin bordism, we will informally discuss topological field theories, their properties, and their relation to Spin bordism.

2.1 Topological Field Theories

We follow [CR18] to develop the general notions of topological field theory. Typically, in quantum field theories on Minkowski space, one works with the **path integral** to compute quantities. The most basic thing that one can compute using a path integral is the **partition function**:

$$Z := \int D\phi e^{iS[\phi]}.$$

This is the integral of the phase $e^{iS[\phi]}$ over the space of all possible fields ϕ on spacetime. The fields ϕ could be a scalar field, vector field, tensor field, combinations of these, etc. The functional $S[\phi]$ is called the **action** of the theory. In many cases, this action is defined from a **Lagrangian density** \mathcal{L} of the fields and their derivatives:

$$S[\phi] = \int d^4x \mathcal{L}.$$

From the path integral, we can also calculate **correlation functions**: If $\mathcal{O}_1, \dots, \mathcal{O}_n$ are observables, i.e. functions of the field ϕ and its derivatives, then their **correlation function** is

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle := \int D\phi \mathcal{O}_1 \dots \mathcal{O}_n e^{iS[\phi]}.$$

Most of the predictions of quantum field theory, i.e. scattering amplitudes and particle content, are then extracted by studying these correlation functions. In practice, the computation of these correlation functions involves two techniques. The first is **perturbation theory**, which involves partially expanding the exponential $e^{iS[\phi]}$ into a Taylor series, truncating at some finite order, and computing the integral of each term that appears.

The second technique, often employed in conjunction with perturbation theory, is **Wick rotation**. This involves extending the time coordinate from a real to a complex number. Then, the Minkowski metric, which has the signature $(n, 1)$ (or $(1, n)$, depending on your metric convention) for real time, becomes a metric of signature $(n, 0)$ (or $(0, n)$) when we restrict the time coordinate to lie on the imaginary axis. Now, there are Wick-rotated versions of the correlation functions and actions, which can be computed as

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle = \int D\phi \mathcal{O}_1 \dots \mathcal{O}_n e^{-S_E[\phi]},$$

and

$$S_E[\phi] = \int d^4x \mathcal{L}_E.$$

Here \mathcal{L}_E is a Wick-rotated form of the Lagrangian density. These correlation functions analytically continue the correlation functions on Minkowski space with respect to the complex time variable. And often computing the correlation functions is much easier in the Euclidean setting, since many of them amount to computing variants of Gaussian integrals. There are additional subtleties to Wick rotation, such as singularities of these functions, but from now on, we will take the Wick-rotated path integral as our foundation. So we will drop the subscript ‘E’ indicating the Euclidean signature.

Now, in topological field theory, we would like to consider a much more general kind of field theory. In particular, we would like interesting topology of spacetime to play a role. So to begin, instead of theories on Euclidean space \mathbb{R}^n , we would like to instead consider theories on a Riemannian manifold M with metric g . There should be no dependence on the metric in a topological field theory, but for now we include it so that we can integrate a Lagrangian density. And while we’re at it, we can let the fields ϕ be general maps to other manifolds. So the correlation functions still have the same formal expression

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle = \int D\phi \mathcal{O}_1 \dots \mathcal{O}_n e^{-S[\phi]},$$

but now ϕ represents a collection of smooth maps from M to other manifolds. Of course, letting these target manifolds be \mathbb{R}^n , we recover the usual notion of a scalar, vector, tensor field, etc. The action is now gotten by integrating the Lagrangian density against the density induced by g , which is $|d^n x| \sqrt{\det g}$ in local coordinates. We say that a field theory on M is a **topological field theory** if the correlation functions do not depend on the metric g .

Now we introduce the functorial picture of topological field theories. In this interpretation of field theory, suppose N is a closed $(n - 1)$ -dimensional manifold. Then our topological field theory should assign a topological vector space of **states** $Z(N)$ to N . Given a compact n -dimensional manifold with boundary M and a diffeomorphism $N_1 \amalg N_2 \rightarrow \partial M$, we should get a map $Z(M) : Z(N_1) \rightarrow Z(N_2)$ encoding the **evolution** of states on N_1 into states on N_2 along M . If M is the cylinder $N \times I$, this should induce the identity map on states $Z(N) \rightarrow Z(N)$ due to topological invariance of the theory. One can think of the cylinder as encoding the imaginary-time evolution of the field theory, but since the length of the cylinder does not matter, evolution over any arbitrarily small amount of imaginary time will give the same result. Finally, composing evolution among multiple n -manifolds should be associative. So, in the language of category theory, a topological field theory determines the data of a functor from the bordism category $\text{Bord}_{\langle n, n-1 \rangle}$ to some category of topological vector spaces.

In the language of the path integral, we can think of a state on N as being a functional on the fields ϕ . This is in analogy to the wave function of a quantum mechanical particle being a function of the particle's configuration. The collection of all these functionals is some topological vector space $Z(N)$. Given a manifold M with $\partial M \cong N$, viewed as a bordism from \emptyset to N , we should obtain an element of $Z(N)$, since \emptyset can only support one state and therefore $Z(\emptyset) \cong \mathbb{C}$. This is given by taking a field configuration ϕ on N to the number

$$Z(M)(\phi) := \int_{\psi|_N = \phi} D\psi e^{-S[\psi]}.$$

Here ψ ranges over the fields on M that restrict to ϕ on N . More generally, suppose we have $(n - 1)$ -manifolds N_1 and N_2 and an n -manifold M with $\partial M \cong N_1 \amalg N_2$. If $\Phi \in Z(N_1)$ is a functional of the fields on N_1 , then we obtain a functional on the fields of N_2 given by

$$Z(M)(\Phi)(\phi_2) := \int_{\psi|_{N_2} = \phi_2} D\psi \Phi(\psi|_{N_1}) e^{-S[\psi]}.$$

However, there is critical aspect of quantum mechanics that we have yet to mention: entanglement. From quantum mechanics, we know that when we put two independent systems together, the resulting system has a Hilbert space that is the tensor product of the two Hilbert spaces of the independent systems. The fact that not all states in the tensor product factorize gives rise to entanglement. So, when we take the disjoint union of two $(n - 1)$ -manifolds N_1 and N_2 , the resulting space of states $Z(N_1 \amalg N_2)$ should be the tensor product of $Z(N_1)$ and $Z(N_2)$, with an appropriate topology. Of course, when the systems evolve separately, the evolution should be given by the tensor product of the separate evolution maps. Therefore we take a topological field theory to be a symmetric monoidal functor from $\text{Bord}_{\langle n, n-1 \rangle}$ (where the monoidal product is disjoint union) to a category of topological vector spaces (where the monoidal product is a tensor product with some suitable topology).

To make this definition as general as we would like, there are a few additional things that we should consider. First, there are additional topological structures on our manifold like orientations, or as we shall see, Spin structures, that we would like to play a role in our field theory. The general treatment of this uses tangential structures:

2.1.1 Tangential structures

Definition 2.1.1. An n -dimensional tangential structure is a fibration $\pi_n : \chi_n \rightarrow BGL_{\mathbb{R}}(n)$. A π_n -structure on an n -manifold M is a homotopy class of maps $M \rightarrow \chi_n$ lifting the map $M \rightarrow BGL_{\mathbb{R}}(n)$ classifying the tangent bundle of M .

In this context, a homotopy is understood to be a homotopy through such lifts. We will often abuse notation by using χ_n to refer to π_n when the map π_n is obvious from the context.

Example 2.1.2. An orientation on M is the same thing as a $BGL_{\mathbb{R}}(n)$ -structure, where π_n is the map $BGL_{\mathbb{R}}(n) \rightarrow BGL_{\mathbb{R}}(n)$ induced by the inclusion $SL_{\mathbb{R}}(n) \rightarrow GL_{\mathbb{R}}(n)$.

Example 2.1.3. A $\text{Spin}_{\mathbb{F}}$ -structure on M is a π_n -structure, where π_n is $B\text{Spin}_{\mathbb{F}}(n) \rightarrow BGL_{\mathbb{R}}(n)$. Here $\text{Spin}(n)$ is the nontrivial double cover of $SO(n)$. For $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, $\text{Spin}_{\mathbb{F}}(n)$ is the quotient of $\text{Spin}(n) \times N_{\mathbb{F}}$ by the $\mathbb{Z}/2\mathbb{Z}$ generated by $(-1, -1)$, where $N_{\mathbb{F}} \subset \mathbb{H}$ is the group of unit elements under multiplication.

In general, if G is a Lie group equipped with a representation $G \rightarrow GL_{\mathbb{R}}(n)$, a G -structure on M is a principal G -bundle P and an isomorphism from $P \times_G \mathbb{R}^n$ to FM , the frame bundle of TM . Here $P \times_G \mathbb{R}^n$ is the **balanced product**, which is the quotient of $P \times \mathbb{R}^n$ under the equivalence relation identifying (pg, v) with (p, gv) for $g \in G$.

By considering $(n-1)$ -manifolds N with π_n -structures on $N \times (-1, 1)$ one can obtain a bordism category of π_n -manifolds, denoted $\text{Bord}_{\langle n, n-1 \rangle}(\pi_n)$, where, loosely speaking, the objects are closed $(n-1)$ -manifolds N with a π_n -structure on $N \times (-1, 1)$ and a morphism from N_1 to N_2 is an equivalence class of compact n -manifolds with π_n -structure giving a bordism from N_1 to N_2 compatible with the π_n -structure. In this way, we can consider topological field theories defined for manifolds with π_n -structures as symmetric monoidal functors from $\text{Bord}_{\langle n, n-1 \rangle}(\pi_n)$ (again with the monoidal structure being given by disjoint union) to a category of topological vector spaces.

The next generalization is to allow the target of the functor defining a topological field theory to be a generic symmetric monoidal category \mathcal{C} . There is nothing profound about this, but it is occasionally useful in more abstract settings. And in the case of extended field theories, it is not always obvious what the n - or (∞, n) -categorical analogue of our category of topological vector spaces should be.

Definition 2.1.4. Let \mathcal{C} be a symmetric monoidal category and let π_n be an n -dimensional tangential structure. An n -dimensional topological field theory with values in \mathcal{C} is a symmetric monoidal functor $Z : \text{Bord}_{\langle n, n-1 \rangle}(\pi_n) \rightarrow \mathcal{C}$.

From here, we can define morphisms of topological field theories, which are symmetric monoidal natural transformations between symmetric monoidal functors. These satisfy the following important property:

Lemma 2.1.5. *All morphisms of topological field theories are isomorphisms.*

Proof. See [CR18, Lemma 2.13] and [CR18, A.2]. □

This follows from the fact that every object in $\text{Bord}_{\langle n, n-1 \rangle}(\pi_n)$ is dualizable, and that a symmetric monoidal natural transformation always gives an isomorphism for dualizable objects.

2.1.2 Invertible Field Theories

So, we have a category of topological field theories. This category admits a symmetric monoidal structure given by applying the monoidal product of \mathcal{C} pointwise. The unit of this structure is the functor $\text{Bord}_{\langle n, n-1 \rangle}(\pi_n) \rightarrow \mathcal{C}$ taking everything to the unit object of \mathcal{C} and its identity map. This allows us to make the following definition:

Definition 2.1.6. An **invertible topological field theory** is a topological field theory that has an inverse with respect to the monoidal product on the category of topological field theories.

If a field theory is invertible, then clearly its value is invertible in \mathcal{C} for every $(n - 1)$ -manifold. In particular, if our field theory takes values in vector spaces, then an invertible field theory associates a line to each $(n - 1)$ -manifold. Also, an invertible field theory assigns invertible maps to all bordisms. The tensor product of two invertible field theories is again invertible, so isomorphism classes of invertible field theories form a group. Or, from a more categorical perspective, invertible field theories and natural isomorphisms between them form a **Picard groupoid**.

Some commentary on this definition is in order. Since quantum systems are projective, an invertible field theory assigns only a single physical state to each $(n - 1)$ -manifold, and therefore the evolution maps are trivial. So how could invertible field theories possibly be interesting?

On the side of physics, a reason one might be interested in invertible field theories is that an invertible field theory might be the low-energy limit of another quantum system. Then the linear data of the morphisms between the lines associated to each $(n - 1)$ -manifold may have significance in the context of the non-invertible theory.

On the side of mathematics, invertible field theories are examples of bordism invariants for manifolds with a given tangential structure, since they assign invertible morphisms to every bordism. If we have an extended field theory (defined below), this means that the invariant is local in some sense. This is very interesting because bordism invariants are typically thought of as global, so local descriptions can be quite interesting.

2.1.3 Extended Field Theories

The final notion that we must introduce is that of an extended field theory. The formal definition of an extended theory is far too complex for the scope of this thesis, since it involves n -categories (or (∞, n) -categories). Using *complete Segal spaces* as a model for (∞, n) -categories, these ideas are developed with much more detail and rigor in [Lur09].

Anyway, we can imagine that there are n - or (∞, n) -categories $\text{Bord}_{\langle n, n-1, \dots, 1, 0 \rangle}(\pi_n)$, where the objects are closed 0-manifolds, the morphisms are compact 1-manifolds with boundary, morphisms between morphisms are compact 2-manifolds with corners, and so on, until we reach compact n -manifolds with corners, all equipped with the relevant notion of a π_n -structure. If we are using n -categories, then these are identified by diffeomorphisms preserving all relevant structure, whereas if we use (∞, n) -categories, the entire space of diffeomorphisms is remembered. With the disjoint union again playing the role of the monoidal product, this category admits a symmetric monoidal structure. We then define a **extended topological field theory** with values in \mathcal{C} to be a symmetric monoidal functor $\text{Bord}_{\langle n, \dots, 0 \rangle}(\pi_n) \rightarrow \mathcal{C}$, where \mathcal{C} is a symmetric monoidal n - or (∞, n) -category, whichever matches our choice for $\text{Bord}_{\langle n, \dots, 0 \rangle}(\pi_n)$.

Extended field theories admit the strongest notion of locality. This is an extremely strong condition. The *cobordism hypothesis*, [Lur09, Theorem 2.4.6], states that for the tangential structure given by framing (trivializations of the tangent bundle), an extended topological field theory is determined by its value on a single point. For other tangential structures, there is a similar result ([Lur09, Theorem 2.4.18]).

Yet again, morphisms between extended field theories are invertible and we can tensor extended field theories, so the above discussion about invertible field theories applies equally well to *extended* invertible field theories.

2.2 The Theories Classified by Spin Bordism

Now that we have the general notion of a topological field theory, we can begin to discuss the theories related to Spin. From now on, we let $\text{Bord}_n(\pi_n) = \text{Bord}_{\langle n, \dots, 0 \rangle}(\pi_n)$ to simplify notation.

Now, following [FH21, Section 5.3], suppose we have an invertible field theory $\text{Bord}_n(\pi_n) \rightarrow \mathcal{C}$. Then this map factors through the maximal groupoid $\mathcal{C}^\times \subseteq \mathcal{C}$ of invertible objects. If we take the classifying spaces of the domain and target of this functor, we get a map of infinite loop spaces $B\text{Bord}_n(\pi_n) \rightarrow B\mathcal{C}^\times$. According to a version of the theorem of Madsen–Galatius–Tillmann–Weiss [Gal+09] adapted for higher bordism categories (see [Lur09, Section 2.5]), the classifying space $B\text{Bord}_n(\pi_n)$ is equivalent to $\Omega^\infty \Sigma^n MT\pi_n$, the infinite loop space of the n -fold suspended *Madsen–Tillmann spectrum* $MT\pi_n$ of the vector bundle on χ_n classified by π_n . The Madsen–Tillmann spectrum is the Thom spectrum of a

vector bundle on χ_n pulled back from the orthogonal complement of the classifying bundle on $BO(n)$.

Taking \mathcal{C} to be an n - or (∞, n) -categorical version of the category of complex vector spaces and passing to spectra, we can define an invertible field theory to be a map of spectra $MT\pi_n \rightarrow IC^\times$, where IC^\times is the Anderson dualizing spectrum of \mathbb{C}^\times . The reason we pass to spectra is that the symmetric monoidal structure on these categories and the fact that topological field theories are symmetric monoidal functors implies that the map of classifying spaces is a morphism of infinite loop spaces. This is motivated by our previous definition and the homotopy types of the classifying spaces, but it is not (yet) known to be equivalent. We make this definition since the IC^\times is characterized by the natural map $[X, IC^\times] \rightarrow \text{Hom}(\pi_0 X, \mathbb{C}^\times)$ being an isomorphism for all spectra X .

Now, to compute groups of deformation classes of invertible field theories, i.e. identifying theories that are connected by a path in the space of theories, it is not appropriate to use the discrete topology of \mathbb{C}^\times . Following [FH21, Definition 5.22], we say that two maps $\alpha_0, \alpha_1 : MT\pi_n \rightarrow IC^\times$ belong to the same **deformation class** if there is an element $\xi \in HC^0 MT\pi_n$ that maps to $\alpha_1 - \alpha_0$ under the map $HC \rightarrow IC^\times$ induced by the exponential map $\mathbb{C} \rightarrow \mathbb{C}^\times$.

Using the fiber sequence $I\mathbb{Z} \rightarrow HC \rightarrow IC^\times$ (here we are identifying \mathbb{Z} with $\mathbb{Z}i \subset \mathbb{C}$, the kernel of \exp), which induces a fiber sequence $HC \rightarrow IC^\times \rightarrow \Sigma I\mathbb{Z}$, we can identify the group of deformation classes of invertible field theories with the torsion subgroup of $[MT\pi_n, \Sigma I\mathbb{Z}]$, since this group is the cokernel of the map $[MT\pi_n, HC] \rightarrow [MT\pi_n, IC^\times]$. To actually compute these groups, one can use the short exact sequences

$$0 \longrightarrow \text{Ext}^1(\pi_n X, \mathbb{Z}) \longrightarrow [X, \Sigma I\mathbb{Z}] \longrightarrow \text{Hom}(\pi_{n+1} X, \mathbb{Z}) \longrightarrow 0$$

for arbitrary spectra X . These sequences split, but not naturally.

Now we turn to Spin structures. Under Wick rotation, the identity component of the Lorentz group corresponds to $SO(n)$. So, we can think of a topological field theory of manifolds with a $\text{Spin}_{\mathbb{F}}$ -structure being a theory of fields with an internal symmetry group isomorphic to the group of elements of unit norm of \mathbb{F} . These arise as symmetry types of relativistic field theories with an internal symmetry group of $N_{\mathbb{F}}$, the group of unit elements of \mathbb{F} . We can think of such a theory as being the Wick-rotated analogue of an extended invertible field theory that is *relativistic* with the fields having an *internal* (i.e. not moving any points of spacetime) action by the group $N_{\mathbb{F}}$, under which the physics is symmetric.

Using the homotopy groups of $M\text{Spin}_{\mathbb{F}} \cong MT\text{Spin}_{\mathbb{F}}$ listed in Table 5.1 and the exact sequence above, we calculate the following table of deformation class groups of fully extended topological field theories of dimension n : These computations appear in [FH21, Section 9.3].

n	$TP_n(\text{Spin})$	$TP_n(\text{Spin}_{\mathbb{C}})$	$TP_n(\text{Spin}_{\mathbb{H}})$
0	0	0	0
1	$\mathbb{Z}/2\mathbb{Z}$	\mathbb{Z}	0
2	$\mathbb{Z}/2\mathbb{Z}$	0	0
3	\mathbb{Z}	\mathbb{Z}^2	$\mathbb{Z}/2\mathbb{Z}^2$
4	0	0	0
5	0	\mathbb{Z}^2	\mathbb{Z}^2
6	0	0	$\mathbb{Z}/2\mathbb{Z}^2$
7	\mathbb{Z}^2	\mathbb{Z}^4	\mathbb{Z}^4
8	0	0	0
9	$\mathbb{Z}/2\mathbb{Z}^2$	\mathbb{Z}^4	$\mathbb{Z}/2\mathbb{Z}$

Table 2.1: Groups of deformation classes of invertible field theories under tensor product.

Chapter 3

Clifford Algebras and Spin Groups

The first characters that we must introduce in our story are the Spin groups and their relatives. The Spin groups are the nontrivial double covers of the special orthogonal groups, and they can be constructed abstractly in this way, but we will construct them using Clifford algebras, since this will give us a deeper understanding of their structure as well as provide the foundation for the Atiyah–Bott–Shapiro map. The definitions we are using, as well as the main theorems we prove, are adapted from [ABS64, Part 1]. For the quaternionic case, our definitions are those of [Hu22].

3.1 The Clifford Algebra Construction

In this section, all vector spaces will be real and finite-dimensional unless otherwise stated.

Definition 3.1.1. Let V be a vector space equipped with a quadratic form Q . The **Clifford algebra** of V , denoted $\text{Cl}(V)$, is the quotient of the tensor algebra generated by V by the two-sided ideal generated by the relations $v^2 = Q(v)$ for all $v \in V$.

Associated to Q is the symmetric bilinear form

$$(v, w) := \frac{1}{2}(Q(v + w) - Q(v) - Q(w)).$$

Applying the defining relations of the Clifford algebra to v , w , and $v + w$, we find that the relations

$$vw + wv = 2(v, w)$$

hold in $\text{Cl}(V)$ for any $v, w \in V$. So the Clifford algebra encodes the geometry of the inner product on V into algebraic relations. This construction can be made functorial, as a map $T : V \rightarrow W$ between inner product spaces such that $(Tv_1, Tv_2) = (v_1, v_2)$ for all $v_1, v_2 \in V$ defines a map $\text{Cl}(V) \rightarrow \text{Cl}(W)$ induced by the map of tensor algebras.

Example 3.1.2. Let $V = \mathbb{R}^n$, let $e_1, \dots, e_n \in V$ be the standard basis vectors, and suppose the quadratic form is induced by the symmetric bilinear form given by $(e_i, e_j) = a_i \delta_{ij}$ for some nonzero $a_i \in \mathbb{R}$. Then $\text{Cl}(V)$ is defined by the relations $e_i e_j + e_j e_i = 2a_i \delta_{ij}$, so $e_i^2 = a_i$. Then $\text{Cl}(V)$ has a basis given by the monomials $e_{i_1} \dots e_{i_k}$, where $i_1 < \dots < i_k$. In particular, the dimension of $\text{Cl}(V)$ is $2^{\dim(V)}$. This can be seen by using the anticommutation relations to reduce a general product of basis vectors in the tensor algebra to linear combinations of these monomials.

If V is a real vector space with a quadratic form, we can form more algebras from $\text{Cl}(V)$ by extending our scalars. Let $\text{Cl}_{\mathbb{C}}(V) = \text{Cl}(V) \otimes_{\mathbb{R}} \mathbb{C}$ and $\text{Cl}_{\mathbb{F}}(V) = \text{Cl}(V) \otimes_{\mathbb{R}} \mathbb{H}$. These are the **complex** and **quaternionic Clifford algebras** of V . As a notational device, we let $\text{Cl}_{\mathbb{R}}(V) = \text{Cl}(V)$, so that we can discuss $\text{Cl}_{\mathbb{F}}(V)$ generally for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

The algebra $\text{Cl}(V)$ can be equipped with an automorphism $\alpha : \text{Cl}(V) \rightarrow \text{Cl}(V)$, induced by the negation map $V \rightarrow V$ taking v to $-v$. Let $\text{Cl}(V)^0 \subseteq \text{Cl}(V)$ be the subspace generated by the images of monomials with an even number of factors, and let $\text{Cl}(V)^1 \subseteq \text{Cl}(V)$ be the subspace generated by the images of monomials with an odd number of factors. Alternatively, $\text{Cl}(V)^0$ is the eigenspace with eigenvalue 1 with respect to α and $\text{Cl}(V)^1$ is the eigenspace with -1 . If we choose a basis of V and examine the basis monomials, it is clear that $\text{Cl}(V) = \text{Cl}(V)^0 \oplus \text{Cl}(V)^1$. The subspaces $\text{Cl}(V)^0$ and $\text{Cl}(V)^1$ make $\text{Cl}(V)$ into a $\mathbb{Z}/2\mathbb{Z}$ -graded algebra, also known as a superalgebra. The automorphism α naturally induces automorphisms $\alpha : \text{Cl}_{\mathbb{F}}(V) \rightarrow \text{Cl}_{\mathbb{F}}(V)$ making $\text{Cl}_{\mathbb{F}}(V)$ into a $\mathbb{Z}/2\mathbb{Z}$ -graded algebra. If $x \in \text{Cl}_{\mathbb{F}}(V)$ is homogeneous (i.e. belongs to $\text{Cl}_{\mathbb{F}}(V)^0$ or $\text{Cl}_{\mathbb{F}}(V)^1$) and nonzero, then we let $|x|$ denote its **grading**, so $|x| = 0$ if $x \in \text{Cl}_{\mathbb{F}}(V)^0$ and $|x| = 1$ if $x \in \text{Cl}_{\mathbb{F}}(V)^1$. If $|x| = 0$, we say the x is **even-graded**, and if $|x| = 1$, we say that x is **odd-graded**.

Proposition 3.1.3. *Suppose V_1 and V_2 are vector spaces with quadratic forms Q_1 and Q_2 . Let Q be the quadratic form on $V = V_1 \oplus V_2$ given by $Q(v_1 + v_2) = Q_1(v_1) + Q_2(v_2)$, where $v_1 \in V_1$ and $v_2 \in V_2$. Then there is an isomorphism of $\mathbb{Z}/2\mathbb{Z}$ -graded algebras $\text{Cl}(V_1) \otimes_{\mathbb{Z}/2\mathbb{Z}} \text{Cl}(V_2) \rightarrow \text{Cl}(V)$ taking $v_1 \otimes v_2$ to $v_1 v_2$.*

Note the $\otimes_{\mathbb{Z}/2\mathbb{Z}}$, which denotes the tensor product of $\mathbb{Z}/2\mathbb{Z}$ -graded \mathbb{R} -algebras. So

$$\begin{aligned} (\text{Cl}(V_1) \otimes_{\mathbb{Z}/2\mathbb{Z}} \text{Cl}(V_2))^0 &= (\text{Cl}(V_1)^0 \otimes \text{Cl}(V_2)^0) \oplus (\text{Cl}(V_1)^1 \otimes \text{Cl}(V_2)^1), \\ (\text{Cl}(V_1) \otimes_{\mathbb{Z}/2\mathbb{Z}} \text{Cl}(V_2))^1 &= (\text{Cl}(V_1)^0 \otimes \text{Cl}(V_2)^1) \oplus (\text{Cl}(V_1)^1 \otimes \text{Cl}(V_2)^0), \end{aligned}$$

and the multiplication for homogeneous elements is given by

$$(x_1 \otimes x_2)(y_1 \otimes y_2) = (-1)^{|x_2||y_1|} (x_1 y_1) \otimes (x_2 y_2).$$

This is the **Koszul sign rule**.

Proof. To see that there is indeed a map of algebras $\text{Cl}(V_1) \otimes_{\mathbb{Z}/2\mathbb{Z}} \text{Cl}(V_2) \rightarrow \text{Cl}(V)$, notice that we get maps of $\mathbb{Z}/2\mathbb{Z}$ -graded algebras $\text{Cl}(V_1) \rightarrow \text{Cl}(V)$ and $\text{Cl}(V_2) \rightarrow \text{Cl}(V)$ induced by the inclusions $V_1 \rightarrow V$ and $V_2 \rightarrow V$. So we get a linear map $\text{Cl}(V_1) \otimes \text{Cl}(V_2) \rightarrow \text{Cl}(V)$ given by tensoring these two maps and multiplying in $\text{Cl}(V)$. By counting the number of factors of a monomial, we see that this is clearly a map of $\mathbb{Z}/2\mathbb{Z}$ -graded vector spaces. So we just need to check that it is a map of algebras. This follows because

$$\begin{aligned} & (e_{i_1} \dots e_{i_a} \otimes e_{j_1} \dots e_{j_b})(e_{k_1} \dots e_{k_c} \otimes e_{\ell_1} \dots e_{\ell_d}) \\ &= (-1)^{bc} e_{i_1} \dots e_{i_a} e_{k_1} \dots e_{k_c} \otimes e_{j_1} \dots e_{j_b} e_{\ell_1} \dots e_{\ell_d} \\ &\mapsto (-1)^{bc} e_{i_1} \dots e_{i_a} e_{k_1} \dots e_{k_c} e_{j_1} \dots e_{j_b} e_{\ell_1} \dots e_{\ell_d} \\ &= (e_{i_1} \dots e_{i_a} e_{j_1} \dots e_{j_b}) \cdot (e_{k_1} \dots e_{k_c} e_{\ell_1} \dots e_{\ell_d}). \end{aligned}$$

This is because the e_{j_1}, \dots, e_{j_b} all anticommute with the e_{k_1}, \dots, e_{k_c} , and commuting $e_{j_1} \dots e_{j_b}$ past $e_{k_1} \dots e_{k_c}$ requires bc transpositions, hence the sign factor $(-1)^{bc}$. So we get a map of $\mathbb{Z}/2\mathbb{Z}$ -graded algebras $\text{Cl}(V_1) \otimes_{\mathbb{Z}/2\mathbb{Z}} \text{Cl}(V_2) \rightarrow \text{Cl}(V)$. Clearly it is surjective, because V in $\text{Cl}(V)$ is generated by the vectors of V_1 and V_2 , and all of these vectors are in the image. Using the formula for the dimension of a tensor product, it follows that this is an isomorphism. \square

There are similar isomorphisms $\text{Cl}_{\mathbb{C}}(V_1) \otimes_{\mathbb{Z}/2\mathbb{Z}} \text{Cl}_{\mathbb{C}}(V_2) \cong \text{Cl}_{\mathbb{C}}(V_1 \oplus V_2)$, where the tensor product is over \mathbb{C} , and $\text{Cl}_{\mathbb{R}}(V_1) \otimes_{\mathbb{Z}/2\mathbb{Z}} \text{Cl}_{\mathbb{H}}(V_2) \cong \text{Cl}_{\mathbb{H}}(V_1 \oplus V_2)$, where the tensor product is over \mathbb{R} .

The algebra $\text{Cl}(V)$ also has an antiautomorphism (linear isomorphism reversing multiplication while preserving the $\mathbb{Z}/2\mathbb{Z}$ -grading) induced by the automorphism of the tensor algebra of V taking $v_1 v_2 \dots v_k$ to $v_k v_{k-1} \dots v_1$. This indeed defines a map $\text{Cl}(V) \rightarrow \text{Cl}(V)$, which we denote by $x \mapsto x^t$ and refer to as the **transpose**, since $v^2 - Q(v)$ maps to itself under the antiautomorphism of the tensor algebra. This operation can be extended to the algebras $\text{Cl}_{\mathbb{F}}(V)$, but we must take care, especially when $\mathbb{F} = \mathbb{H}$, since the quaternions do not commute. So, we extend $(-)^t$ to $\text{Cl}_{\mathbb{F}}(V)$ by tensoring with the map $a \mapsto \bar{a}$ on scalars, which is an antiautomorphism of \mathbb{F} . We can combine α and $(-)^t$ to form another antiautomorphism of $\text{Cl}_{\mathbb{F}}(V)$, denoted $\bar{x} = \alpha(x^t) = \alpha(x)^t$. Finally, we can define a function N on $\text{Cl}_{\mathbb{F}}(V)$ by setting $N(x) = x\bar{x}$. Note that $\overline{N(x)} = N(x)$. This function takes values in $\text{Cl}_{\mathbb{F}}(V)$, but can be thought of as a sort of norm.

Using the classification of inner products on finite-dimensional real vector spaces, we see that there is essentially no loss of generality in choosing a standard model for the vector space generating a Clifford algebra for each dimension (for negative-definite quadratic forms, which will be the ones we are interested in). So, we equip \mathbb{R}^n with the quadratic form induced by the negative of the Euclidean inner product, and define $\text{Cl}_{\mathbb{F}}(n) = \text{Cl}_{\mathbb{F}}(\mathbb{R}^n)$. There are inclusions $\text{Cl}_{\mathbb{F}}(n) \rightarrow \text{Cl}_{\mathbb{F}}(n+1)$ induced by the maps $\mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$ given by inserting a zero at the end of the tuple. Note that these maps are not canonical, because our choice of inclusion $\mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$ was not canonical. However, any two maps $\text{Cl}_{\mathbb{F}}(n) \rightarrow \text{Cl}_{\mathbb{F}}(n+1)$ induced by an

inclusion $\mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$ differ by an automorphism of $\text{Cl}_{\mathbb{F}}(n+1)$ induced by an automorphism of \mathbb{R}^{n+1} .

3.2 Clifford Modules

The main way we will understand Clifford algebras is by identifying the modules they support. We will only be interested in left modules, so if all modules will be left modules if unspecified. To do this, we can identify the algebras $\text{Cl}_{\mathbb{F}}(n)$ with matrix algebras. While this discards the structure of the $\mathbb{Z}/2\mathbb{Z}$ -grading, we will see that this can be circumvented. To do this, we introduce the algebras $\text{Cl}^+(n)$, which are defined as $\text{Cl}(\mathbb{R}^n)$, but here \mathbb{R}^n has the *positive definite* quadratic form induced by the Euclidean inner product.

Lemma 3.2.1. *There are isomorphisms*

$$\text{Cl}(n) \otimes_{\mathbb{R}} \text{Cl}^+(2) \cong \text{Cl}^+(n+2), \quad (3.1)$$

$$\text{Cl}^+(n) \otimes_{\mathbb{R}} \text{Cl}(2) \cong \text{Cl}(n+2). \quad (3.2)$$

Proof. Define a map $\phi : \mathbb{R}^{n+2} \rightarrow \text{Cl}^+(n) \otimes_{\mathbb{R}} \text{Cl}(2)$ by

$$\phi(e_i) := \begin{cases} 1 \otimes e_i & i \in \{1, 2\}, \\ e_i \otimes e_1 e_2 & i \notin \{1, 2\}. \end{cases}$$

This extends to an algebra homomorphism $\phi : \text{Cl}(n+2) \rightarrow \text{Cl}^+(n) \otimes_{\mathbb{R}} \text{Cl}(2)$, since $\phi(e_i)\phi(e_j) + \phi(e_j)\phi(e_i) = -2\delta_{ij}$. We see that the image of ϕ contains all the generators of $\text{Cl}(n) \otimes_{\mathbb{R}} \text{Cl}^+(2)$, since we have $e_i \otimes 1 = \phi(e_i)\phi(e_2)\phi(e_1)$. So ϕ is surjective, and by comparing the dimension of $\text{Cl}^+(n+2)$ and $\text{Cl}(n) \otimes_{\mathbb{R}} \text{Cl}^+(2)$, we see that it must be an isomorphism.

For the other isomorphism, we can define a map $\phi^+ : \mathbb{R}^{n+2} \rightarrow \text{Cl}(n) \otimes_{\mathbb{R}} \text{Cl}^+(2)$ by

$$\phi^+(e_i) := \begin{cases} 1 \otimes e_i & i \in \{1, 2\}, \\ e_i \otimes e_1 e_2 & i \notin \{1, 2\}. \end{cases}$$

As in the previous case, this extends to an algebra homomorphism $\phi^+ : \text{Cl}^+(n+2) \rightarrow \text{Cl}(n) \otimes_{\mathbb{R}} \text{Cl}^+(2)$, and it is an isomorphism for the same reasons. \square

So if we can compute the isomorphism type of $\text{Cl}(n)$ and $\text{Cl}^+(n)$ for $n = 1, 2$, then we can extrapolate the isomorphism type of $\text{Cl}_{\mathbb{F}}(n)$ for all n . Let $\mathbb{F}(n)$ denote the algebra of $n \times n$ matrices over \mathbb{F} .

Lemma 3.2.2. *We have the following identifications:*

$$\text{Cl}(1) \cong \mathbb{C},$$

$$\begin{aligned}
\text{Cl}^+(1) &\cong \mathbb{R}^2, \\
\text{Cl}(2) &\cong \mathbb{H}, \\
\text{Cl}^+(2) &\cong \mathbb{R}(2).
\end{aligned}$$

Proof. For $\text{Cl}(1)$, we have $e_1^2 = -1$, so $\text{Cl}(1) \cong \mathbb{C}$. For $\text{Cl}^+(1)$, we have $e_1^2 = 1$, so we can identify $\text{Cl}^+(1) \cong \mathbb{R}^2$, where the two factors are generated by $\frac{1}{2}(1 + e_1)$ and $\frac{1}{2}(1 - e_1)$. For $\text{Cl}(2)$, we can identify this algebra with the quaternions \mathbb{H} , with $e_1 \mapsto i$ and $e_2 \mapsto j$. Finally, $\text{Cl}^+(2)$ can be identified with $\mathbb{R}(2)$, where $e_1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $e_2 \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. \square

Then to compute the isomorphism classes of $\text{Cl}_{\mathbb{F}}(n)$, we can use the following common identities:

Lemma 3.2.3. *There are isomorphisms of algebras*

$$\begin{aligned}
\mathbb{R}(n) \otimes_{\mathbb{R}} \mathbb{F} &\cong \mathbb{F}(n), \\
\mathbb{R}(n) \otimes_{\mathbb{R}} \mathbb{R}(m) &\cong \mathbb{R}(nm), \\
\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} &\cong \mathbb{C}^2, \\
\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} &\cong \mathbb{C}(2), \\
\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} &\cong \mathbb{R}(4).
\end{aligned}$$

Applying these isomorphisms and using our computations of $\text{Cl}(n)$ and $\text{Cl}^+(n)$ for $n = 1, 2$, we get the following determination of $\text{Cl}_{\mathbb{F}}(n)$:

Theorem 3.2.4. *The algebras $\text{Cl}_{\mathbb{F}}(n)$ are isomorphic to the following algebras:*

n	$\text{Cl}(n)$	$\text{Cl}^+(n)$	$\text{Cl}_{\mathbb{C}}(n)$	$\text{Cl}_{\mathbb{H}}(n)$
0	\mathbb{R}	\mathbb{R}	\mathbb{C}	\mathbb{H}
1	\mathbb{C}	\mathbb{R}^2	\mathbb{C}^2	$\mathbb{C}(2)$
2	\mathbb{H}	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{R}(4)$
3	\mathbb{H}^2	$\mathbb{C}(2)$	$\mathbb{C}(2)^2$	$\mathbb{R}(4)^2$
4	$\mathbb{H}(2)$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$
5	$\mathbb{C}(4)$	$\mathbb{H}(2)^2$	$\mathbb{C}(4)^2$	$\mathbb{C}(8)$
6	$\mathbb{R}(8)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{H}(8)$
7	$\mathbb{R}(8)^2$	$\mathbb{C}(8)$	$\mathbb{C}(8)^2$	$\mathbb{H}(8)^2$
8	$\mathbb{R}(16)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$

Moreover, there exist isomorphisms

$$\begin{aligned}
\text{Cl}(n + 8) &\cong \text{Cl}(n) \otimes_{\mathbb{R}} \mathbb{R}(16), \\
\text{Cl}_{\mathbb{C}}(n + 2) &\cong \text{Cl}_{\mathbb{C}}(n) \otimes_{\mathbb{C}} \mathbb{C}(2), \\
\text{Cl}_{\mathbb{H}}(n + 8) &\cong \text{Cl}_{\mathbb{H}}(n) \otimes_{\mathbb{R}} \mathbb{H}(16), \\
\text{Cl}_{\mathbb{H}}(n + 4) &\cong \text{Cl}(n) \otimes_{\mathbb{R}} \mathbb{R}(8).
\end{aligned}$$

Proof. The table can be filled out routinely using the lemmas above. The “periodicity” isomorphisms follow by calculating the relevant tensor products of $\text{Cl}(2)$, $\text{Cl}^+(2)$, and \mathbb{F} . \square

Already, we can see the manifestation of Bott periodicity. Real and quaternionic Clifford algebras exhibit a periodicity of period eight, and complex Clifford algebras have period two. Moreover, the quaternionic Clifford algebras are related to the real Clifford algebras with a shift of degree four.

The isomorphisms above give us a way to completely classify ungraded modules. The graded structure appears to be completely opaque in the matrix algebras. However, we have a trick to extract graded modules from ungraded ones.

Lemma 3.2.5. *There are isomorphisms of algebras $\text{Cl}_{\mathbb{F}}(n+1)^0 \cong \text{Cl}_{\mathbb{F}}(n)$.*

Proof. Define a linear map $\phi : \mathbb{R}^n \rightarrow \text{Cl}_{\mathbb{F}}(n+1)^0$ by setting $\phi(e_i) = e_i e_{n+1}$. Then one can quickly verify $\phi(e_i)\phi(e_j) + \phi(e_j)\phi(e_i) = -2\delta_{ij}$, so ϕ extends to a map of algebras $\phi : \text{Cl}_{\mathbb{F}}(n) \rightarrow \text{Cl}_{\mathbb{F}}(n+1)^0$. Then all monomials in the basis for $\text{Cl}_{\mathbb{F}}(n+1)^0$ induced by the basis $e_1, \dots, e_{n+1} \in \mathbb{R}^{n+1}$ can be produced by forming products of the $\phi(e_i) = e_i e_{n+1}$. Comparing the dimension of $\text{Cl}_{\mathbb{F}}(n)$ and $\text{Cl}_{\mathbb{F}}(n+1)^0$, we see that this must be an isomorphism. \square

Lemma 3.2.6. *If V is not trivial, then the category of graded left $\text{Cl}_{\mathbb{F}}(V)$ -modules is equivalent to the category of left $\text{Cl}_{\mathbb{F}}(V)^0$ -modules.*

Proof. Let \mathcal{C} be the category of graded left $\text{Cl}_{\mathbb{F}}(V)$ -modules and let \mathcal{C}^0 be the category of left $\text{Cl}_{\mathbb{F}}(V)^0$ -modules. We have a functor $F : \mathcal{C} \rightarrow \mathcal{C}^0$ that takes a graded $\text{Cl}_{\mathbb{F}}(V)$ -module M to M^0 . And we let $G : \mathcal{C}^0 \rightarrow \mathcal{C}$ be the functor that takes a left $\text{Cl}_{\mathbb{F}}(V)^0$ -module N to $G(N) = \text{Cl}_{\mathbb{F}}(V) \otimes_{\text{Cl}_{\mathbb{F}}(V)^0} N$, where the grading on $G(N)$ is induced by the grading on $\text{Cl}_{\mathbb{F}}(V)$. There is a natural isomorphism from FG to the identity functor of \mathcal{C}^0 , since the 0-graded piece of $\text{Cl}_{\mathbb{F}}(V) \otimes_{\text{Cl}_{\mathbb{F}}(V)^0} N$ is $\text{Cl}_{\mathbb{F}}(V)^0 \otimes_{\text{Cl}_{\mathbb{F}}(V)^0} N \cong N$. The natural isomorphism GF to the identity functor of \mathcal{C} is given by sending $x \otimes m$ in $\text{Cl}_{\mathbb{F}}(V) \otimes_{\text{Cl}_{\mathbb{F}}(V)^0} M^0$ to $xm \in M$. This is an isomorphism because we have an inverse map $M \rightarrow GF(M)$ given by $m \mapsto 1 \otimes m$ if $m \in M^0$, and $m \mapsto \frac{1}{Q(v)}v \otimes vm$ if $m \in M^1$, where $v \in V$ is some fixed nonzero vector. \square

It is easy to classify left $\text{Cl}_{\mathbb{F}}(n)$ -modules, since $\text{Cl}_{\mathbb{F}}(n)$ is isomorphic to a product of matrix algebras, and finitely generated modules over $\mathbb{F}(n)$ are direct sums of \mathbb{F}^n (see [Lan02, Theorem XVII.5.5]). So we can easily identify the 0-graded part of the graded modules over $\text{Cl}_{\mathbb{F}}(n)$.

In particular, for values of n such that $\text{Cl}(n-1)$ is isomorphic to a matrix algebra, there is just one irreducible graded $\text{Cl}(n)$ -module. But if $\text{Cl}(n-1)$ is a product of two matrix algebras, then there are two irreducible graded $\text{Cl}(n)$ -modules. In the case where there are two, we can understand the relationship between these modules quite well. Graded $\text{Cl}_{\mathbb{F}}(V)$ -modules, as do graded modules over any $\mathbb{Z}/2\mathbb{Z}$ -graded algebra, admit an operation

(endofunctor) denoted $(-)_*$, where $M_*^i = M^{i+1}$. Clearly this operation is an equivalence, and it is its own inverse. This is the **grading involution**.

Since our general computational device with graded Clifford modules over $\text{Cl}_{\mathbb{F}}(n)$ is to pass to ungraded modules over $\text{Cl}_{\mathbb{F}}(n-1)$, we should examine what this involution does for these modules.

Lemma 3.2.7. *Under the equivalence of graded left $\text{Cl}_{\mathbb{F}}(n)$ -modules and left $\text{Cl}_{\mathbb{F}}(n-1)$ -modules, the involution $(-)_*$ corresponds to the action of the automorphism α of $\text{Cl}_{\mathbb{F}}(n-1)$.*

Proof. First, consider the inner automorphism of $\text{Cl}_{\mathbb{F}}(n)$ induced by e_n . Then this automorphism induces $(-)_*$ on graded left $\text{Cl}_{\mathbb{F}}(n)$ -modules, since if we let $\psi_M : M_* \rightarrow M_{e_n}$ (M_{e_n} is M considered with the action induced by the inner automorphism given by e_n) be the map given by $\psi_M(m) = e_n m$, this is a natural isomorphism. So, taking the zero graded part, we see that $(-)_*$ corresponds to the action of the automorphism induced by e_n on the left $\text{Cl}_{\mathbb{F}}(n)^0$ -modules.

Under the isomorphism $\phi : \text{Cl}_{\mathbb{F}}(n) \rightarrow \text{Cl}_{\mathbb{F}}(n+1)^0$, the automorphism induced by e_n on $\text{Cl}_{\mathbb{F}}(n+1)^0$ corresponds to the automorphism α of $\text{Cl}_{\mathbb{F}}(n)$, since $\phi(e_i) = e_i e_n$, which goes to $e_n e_i e_n e_n^{-1} = e_n e_i = -e_i e_n = \phi(-e_i)$ under the automorphism. \square

Now, we are able to see what happens to the irreducible modules.

Proposition 3.2.8. *Suppose that $\text{Cl}_{\mathbb{F}}(n)$ is such that $\text{Cl}_{\mathbb{F}}(n-1)$ is a product of two matrix algebras. Then the operation $(-)_*$ exchanges the isomorphism classes of the two distinct irreducible graded $\text{Cl}_{\mathbb{F}}(n)$ -modules.*

Proof. Using the previous lemma, it instead suffices to consider irreducible $\text{Cl}_{\mathbb{F}}(n-1)$ -modules and how they behave with respect to the automorphism α of $\text{Cl}_{\mathbb{F}}(n-1)$. Now, for any \mathbb{F}_0 , the only two-sided ideals of $\mathbb{F}_0(k)$ are 0 and $\mathbb{F}_0(k)$ by [Lan02, Theorem XVII.5.2]. So, $\text{Cl}_{\mathbb{F}}(n-1) \cong \mathbb{F}_1(n_1) \times \mathbb{F}_2(n_2)$ has only two two-sided ideals that are not zero or the whole algebra: the two summands.¹

We claim that the automorphism α interchanges these two summands. By Theorem 3.2.4, we see that $n = 4m$ if $\mathbb{F} = \mathbb{R}, \mathbb{H}$ and $n = 2m$ if $\mathbb{F} = \mathbb{C}$.

- If $\mathbb{F} = \mathbb{R}, \mathbb{H}$ and $n = 4m$, let $\omega := e_1 e_2 \dots e_{4m-1}$. This commutes with all the e_i , since if we multiply on the left, we do $i-1$ anticommutations until we can apply $e_i^2 = -1$, and if we multiply on the right we do $4m-1-i$ anticommutations until we can apply $e_i^2 = -1$, and so we get the same sign. Hence ω is a central element. We also compute $\omega^2 = 1$, since ω^2 is $e_1^2 e_2^2 \dots e_{4m-1}^2 = -1$ up to a sign factor of $(4m-2) + (4m-3) + \dots + 1 = 1 \pmod{2}$.

¹Here, and in the rest of this thesis, the subscript i in \mathbb{F}_i is merely an index, so \mathbb{F}_i is still assumed to take one of the values $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and not denote a finite field.

- If $\mathbb{F} = \mathbb{C}$ and $n = 2m$, let $\omega := i^m e_1 e_2 \dots e_{2m-1}$. This is central for the same reasons as above. And $\omega^2 = 1$, since ω^2 is $i^{2m} e_1^2 e_2^2 \dots e_{2m-1}^2 = (-1)^{m+1}$ up to a sign factor of $(2m-2) + (2m-3) + \dots + 1 = m-1 \pmod{2}$.

In any case, let $I_{\pm} = (1 \pm \omega) \text{Cl}_{\mathbb{F}}(n-1)$. This is clearly a two-sided ideal because 1 and ω are central. Obviously these are not trivial, and they are not the entire algebra either, since if $1 = (1 \pm \omega)a$ for some a , then we can multiply by ω to get $\omega = (\omega \pm 1)a = \pm(1 \pm \omega)a = \pm 1$, a contradiction. And these ideals are distinct, because if we had $1 + \omega = (1 - \omega)a$ for some a , then multiplying by ω would give us $\omega + 1 = (\omega - 1)a = -1 - \omega$, and therefore $\omega = -1$, again a contradiction. So these two ideals must be the two matrix algebra summands of $\text{Cl}_{\mathbb{F}}(n-1)$. And $\alpha(1) = 1$ and $\alpha(\omega) = -\omega$, so the automorphism α swaps these ideals.

Now we can easily see that the action of α interchanges the two irreducible $\text{Cl}_{\mathbb{F}}(n-1)$ -modules, since the irreducible modules are $\mathbb{F}_i^{n_i}$ with the action given by projection onto $\mathbb{F}_i(n_i)$ and the matrix on vector action. Since α swaps these two factors of $\text{Cl}_{\mathbb{F}}(n-1)$, it swaps the irreducible modules. \square

Much of the interesting structure of Clifford modules is the way they can be tensored together.

Definition 3.2.9. A triple of fields $(\mathbb{F}_1, \mathbb{F}_2, \mathbb{F})$ is **compatible** if it is one of the following: $(\mathbb{R}, \mathbb{R}, \mathbb{R})$, $(\mathbb{C}, \mathbb{C}, \mathbb{C})$, or $(\mathbb{R}, \mathbb{H}, \mathbb{H})$.

Loosely, if $(\mathbb{F}_1, \mathbb{F}_2, \mathbb{F})$ is a compatible triple, then we can tensor something over \mathbb{F}_1 and something over \mathbb{F}_2 to get something over \mathbb{F} . For example, we have an isomorphism $\text{Cl}_{\mathbb{F}_1}(n_1) \otimes_{\mathbb{Z}/2\mathbb{Z}} \text{Cl}_{\mathbb{F}_2}(n_2) \rightarrow \text{Cl}_{\mathbb{F}}(n_1 + n_2)$ induced by $\mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2} \cong \mathbb{R}^{n_1+n_2}$ (here the graded tensor product is taken over \mathbb{F}_1).

Note that we could also consider $(\mathbb{R}, \mathbb{C}, \mathbb{C})$ or even $(\mathbb{H}, \mathbb{H}, \mathbb{R})$ to be compatible, but these are not needed.

So, if M_i is a graded $\text{Cl}_{\mathbb{F}_i}(n_i)$ -module for $i = 1, 2$, then we get a graded $\text{Cl}_{\mathbb{F}_1}(n_1) \otimes_{\mathbb{Z}/2\mathbb{Z}} \text{Cl}_{\mathbb{F}_2}(n_2)$ -module $M_1 \otimes_{\mathbb{Z}/2\mathbb{Z}} M_2$. As with algebras, the grading is given by

$$\begin{aligned} (M_1 \otimes_{\mathbb{Z}/2\mathbb{Z}} M_2)^0 &:= (M_1^0 \otimes M_2^0) \oplus (M_1^1 \otimes M_2^1), \\ (M_1 \otimes_{\mathbb{Z}/2\mathbb{Z}} M_2)^1 &:= (M_1^0 \otimes M_2^1) \oplus (M_1^1 \otimes M_2^0) \end{aligned}$$

and the multiplication again is defined using the Koszul sign rule:

$$(x_1 \otimes x_2)(m_1 \otimes m_2) := (-1)^{|x_2||m_1|} (x_1 m_1) \otimes (x_2 m_2).$$

Tensor products interact nicely with the grading involution:

Lemma 3.2.10. *If M_i is a graded $\text{Cl}_{\mathbb{F}_i}(n_i)$ -module, then there are natural isomorphisms*

$$(M_1 \otimes_{\mathbb{Z}/2\mathbb{Z}} M_2)_* \cong (M_1)_* \otimes_{\mathbb{Z}/2\mathbb{Z}} M_2 \cong M_1 \otimes_{\mathbb{Z}/2\mathbb{Z}} (M_2)_*$$

of graded $\text{Cl}_{\mathbb{F}}(n_1 + n_2) \cong \text{Cl}_{\mathbb{F}_1}(n_1) \otimes_{\mathbb{Z}/2\mathbb{Z}} \text{Cl}_{\mathbb{F}_2}(n_2)$ -modules.

These products are also associative in a certain sense. Suppose we have a “compatible quadruple” $(\mathbb{F}_1, \mathbb{F}_2, \mathbb{F}_3, \mathbb{F})$, which is equal to $(\mathbb{R}, \mathbb{R}, \mathbb{R}, \mathbb{R})$, $(\mathbb{C}, \mathbb{C}, \mathbb{C}, \mathbb{C})$, or $(\mathbb{R}, \mathbb{R}, \mathbb{H}, \mathbb{H})$. We need to specify how we identify $\mathbb{R}^a \oplus \mathbb{R}^b$ with \mathbb{R}^{a+b} . We use the map $\mathbb{R}^a \oplus \mathbb{R}^b \rightarrow \mathbb{R}^{a+b}$ sending $e_i \in \mathbb{R}^a$ to $e_i \in \mathbb{R}^{a+b}$ and $e_i \in \mathbb{R}^b$ to $e_{a+i} \in \mathbb{R}^{a+b}$. Then the diagram

$$\begin{array}{ccc}
 (\mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}) \oplus \mathbb{R}^{n_3} & \xrightarrow{\quad\quad\quad} & \mathbb{R}^{n_1} \oplus (\mathbb{R}^{n_2} \oplus \mathbb{R}^{n_3}) \\
 \downarrow & & \downarrow \\
 \mathbb{R}^{n_1+n_2} \oplus \mathbb{R}^{n_3} & & \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2+n_3} \\
 \searrow & & \swarrow \\
 & \mathbb{R}^{n_1+n_2+n_3} &
 \end{array}$$

commutes, where the horizontal arrow is the standard associator and all others are induced by our choice of identification. It is then easily seen that the corresponding diagram of tensor products of the Clifford algebras $\text{Cl}_{\mathbb{F}_i}(n_i)$ and isomorphisms commutes. From this, we see if M_i is a graded $\text{Cl}_{\mathbb{F}_i}(n_i)$ -module, then there is a natural isomorphism $(M_1 \otimes_{\mathbb{Z}/2\mathbb{Z}} M_2) \otimes_{\mathbb{Z}/2\mathbb{Z}} M_3 \cong M_1 \otimes_{\mathbb{Z}/2\mathbb{Z}} (M_2 \otimes_{\mathbb{Z}/2\mathbb{Z}} M_3)$ of graded $\text{Cl}_{\mathbb{F}}(n_1 + n_2 + n_3)$ -modules.

The remaining properties of Clifford modules that we need to discuss is their behavior under restriction along the maps $\text{Cl}_{\mathbb{F}}(n) \rightarrow \text{Cl}_{\mathbb{F}}(n+1)$. However, these results are best stated in the formalism of K-theory, and so we defer it until the next chapter.

3.3 Spin Groups

We can now go ahead and define Spin groups using these Clifford algebras. The idea is to isolate a subgroup of the group of units in $\text{Cl}(V)$ that we can naturally map to the special orthogonal group of V . So, we begin by defining the **Clifford group** $\Gamma_{\mathbb{F}}(V)$ of $\text{Cl}_{\mathbb{F}}(V)$ to be the set of invertible elements $x \in \text{Cl}_{\mathbb{F}}(V)$ such that $\alpha(x)v x^{-1} \in V$ for all $v \in V$, where α is the grading operator that acts by the identity on $\text{Cl}(V)^0$ and by negation on $\text{Cl}(V)^1$. This group naturally has a representation acting on V , where $x \in \Gamma_{\mathbb{F}}(V)$ acts on $v \in V$ by $xv = \alpha(x)v x^{-1}$.

Lemma 3.3.1. *The Clifford group $\Gamma_{\mathbb{F}}(V)$ has the following properties:*

1. $\Gamma_{\mathbb{F}}(V)$ is a Lie group.
2. The operations $x \mapsto \alpha(x)$, $x \mapsto (x)^t$, and $x \mapsto \bar{x}$ preserve $\Gamma_{\mathbb{F}}(V)$.
3. If $x \in \Gamma_{\mathbb{F}}(V)$ acts trivially on V , then $x \in \mathbb{F}^{\times}$.

4. If $x \in \Gamma_{\mathbb{F}}(V)$, then $N(x) \in \mathbb{R}^{\times}$.
5. The map $N : \Gamma_{\mathbb{F}}(V) \rightarrow \mathbb{R}^{\times}$ is a homomorphism.
6. The representation of $\Gamma_{\mathbb{F}}(V)$ on V acts by orthogonal transformations.

Proof. 1. Let n be the dimension of V . Since $\text{Cl}_{\mathbb{F}}(V)$ is a finite-dimensional \mathbb{R} -algebra of dimension $k = \dim(\mathbb{F}) \cdot 2^n$, we equip it with the standard smooth structure of \mathbb{R}^k and then its group of units is a Lie group that is an open submanifold of $\text{Cl}_{\mathbb{F}}(V)$. Then we have a map $f : \text{Cl}_{\mathbb{F}}(V)^{\times} \rightarrow V \otimes_{\mathbb{R}} (\text{Cl}_{\mathbb{F}}(V)/V)$ given by choosing a basis e_1, \dots, e_n for V and sending $x \in \text{Cl}_{\mathbb{F}}(V)^{\times}$ to $e_1 \otimes \alpha(x)e_1x^{-1} + \dots + e_n \otimes \alpha(x)e_nx^{-1}$. Clearly $\Gamma_{\mathbb{F}}(V) = f^{-1}(0)$, so $\Gamma_{\mathbb{F}}(V) \subseteq \text{Cl}_{\mathbb{F}}(V)^{\times}$ is a closed subgroup. By the closed subgroup theorem (see [Lee13, Theorem 20.12]), we see that $\Gamma_{\mathbb{F}}(V)$ is a Lie group.

2. The operation $x \mapsto \alpha(x)$ is an automorphism of $\text{Cl}_{\mathbb{F}}(V)$ that restricts to an automorphism of V . Note that α^2 is the identity. So if $x \in \Gamma_{\mathbb{F}}(V)$ and $v \in V$, then $\alpha^{-1}(v) \in V$, so $\alpha(x)\alpha^{-1}(v)x^{-1} \in V$, and therefore $\alpha(\alpha(x))v\alpha(x)^{-1} \in V$, which implies $\alpha(x) \in \Gamma_{\mathbb{F}}(V)$. The operations $x \mapsto (x)^t$ and $x \mapsto \bar{x}$ are antiautomorphisms of $\text{Cl}_{\mathbb{F}}(V)$ that restrict to automorphisms of V and commute with α . Let β denote one of these operations. Then $x^{-1}\beta^{-1}(v)\alpha(x) \in V$, so $\beta(\alpha(x))v\beta(x)^{-1} = \alpha(\beta(x))v\beta(x)^{-1} \in V$, and therefore $\beta(x) \in \Gamma_{\mathbb{F}}(V)$.
3. Suppose $x \in \Gamma_{\mathbb{F}}(V)$ acts trivially on V . Choose an orthogonal basis e_1, \dots, e_n for V . Then using the monomial basis for $\text{Cl}_{\mathbb{F}}(V)$ that this orthogonal basis determines, for each $i \in \{1, \dots, n\}$, we can write $x = a_{i,0} + a_{i,1} + b_{i,0}e_i + b_{i,1}e_i$ where $a_{i,j}$ and $b_{i,j}$, with $a_{i,j}, b_{i,j} \in \text{Cl}_{\mathbb{F}}(V)^j$, do not contain any factors of e_i when expressed in the monomial basis. Since $\alpha(x)e_ix^{-1} = e_i$, we have

$$\begin{aligned} a_{i,0}e_i - a_{i,1}e_i - b_{i,0}e_i^2 + b_{i,1}e_i^2 &= e_ia_{i,0} + e_ia_{i,1} + e_ib_{i,0}e_i + e_ib_{i,1}e_i \\ &= a_{i,0}e_i - a_{i,1}e_i + b_{i,0}e_i^2 - b_{i,1}e_i^2. \end{aligned}$$

It follows that $b_{i,0} = b_{i,1} = 0$, so that x does not have any monomials containing e_i . Hence $x \in \mathbb{F}$. But x is a unit, so $x \in \mathbb{F}^{\times}$.

4. Suppose $x \in \Gamma_{\mathbb{F}}(V)$. Then $\alpha(x^t) \in \Gamma_{\mathbb{F}}(V)$ so if $v \in V$, we have $x^tv\alpha(x^t)^{-1} \in V$. Because $v^t = v$ for $v \in V$, we have $x^tv\alpha(x^t)^{-1} = (\alpha(x^t)^{-1})^tvx = \alpha(x)^{-1}vx$. So $v = \alpha(x)x^tv\alpha(x^t)^{-1}x^{-1}$. But $\alpha(x)x^tv\alpha(x^t)^{-1}x^{-1} = \alpha(x\alpha(x^t))v(x\alpha(x^t))^{-1}$ so $x\alpha(x^t) = x\bar{x} = N(x) \in \mathbb{F}^{\times}$. But since $N(x) = \overline{N(x)}$, $N(x) \in \mathbb{R}^{\times}$.
5. Suppose $x, y \in \Gamma_{\mathbb{F}}(V)$. Then

$$N(xy) = xy\overline{xy}$$

$$\begin{aligned}
&= xN(y)\bar{x} \\
&= x\bar{x}N(y) \\
&= N(x)N(y).
\end{aligned}$$

6. Note that if $v \in V$, then $N(v) = -v^2 = -Q(v)$. Then if $x \in \Gamma_{\mathbb{F}}(V)$, we have

$$\begin{aligned}
N(\alpha(x)vx^{-1}) &= \alpha(x)vx^{-1}\overline{x^{-1}v}x \\
&= N(x^{-1})N(v)\alpha(x)x \\
&= N(x)^{-1}N(v)\alpha(x\bar{x}) \\
&= N(v).
\end{aligned}$$

So x preserves the quadratic form Q and therefore the inner product of V . □

Definition 3.3.2. $\text{Pin}_{\mathbb{F}}(V)$ is the kernel of $N : \Gamma_{\mathbb{F}}(V) \rightarrow \mathbb{F}^{\times}$.

Clearly $\text{Pin}_{\mathbb{F}}(V)$ is a closed Lie subgroup of $\Gamma_{\mathbb{F}}(V)$. Now we assume that Q is a definite quadratic form.

Proposition 3.3.3. *Let $N_{\mathbb{F}} \subseteq \mathbb{F}^{\times}$ be the subgroup of elements of unit norm. The map $\text{Pin}_{\mathbb{F}}(V) \rightarrow \text{O}(V)$ sits in a short exact sequence*

$$1 \longrightarrow N_{\mathbb{F}} \longrightarrow \text{Pin}_{\mathbb{F}}(V) \longrightarrow \text{O}(V) \longrightarrow 1$$

Proof. As we saw previously, the kernel of $\Gamma_{\mathbb{F}}(V) \rightarrow \text{O}(V)$ is \mathbb{F}^{\times} , and the intersection of \mathbb{F}^{\times} with $\text{Pin}_{\mathbb{F}}(V)$ is $N_{\mathbb{F}}$. So we just need to show that $\text{Pin}_{\mathbb{F}}(V) \rightarrow \text{O}(V)$ is surjective. This is true because $\text{O}(V)$ is generated by reflections. A reflection through the hyperplane orthogonal to a unit vector $w \in V$ is given by the linear map $v \mapsto v - 2Q(w)(w, v)w$. If we let $x = w$, then $N(w) = -Q(w) = \pm 1$ and

$$\begin{aligned}
\alpha(w)vw^{-1} &= (-w)v(Q(w)w) \\
&= -Q(w)(2(w, v) - vw)w \\
&= v - 2Q(w)(w, v)w.
\end{aligned}$$

So all the reflections are in the image, and therefore the map is surjective. □

Finally, we are in a position to define the Spin groups.

Definition 3.3.4. The group $\text{Spin}_{\mathbb{F}}(V)$ is the preimage of $\text{SO}(V)$ under the map $\text{Pin}_{\mathbb{F}}(V) \rightarrow \text{O}(V)$.

Clearly $\text{Spin}_{\mathbb{F}}(V)$ is an open Lie subgroup of $\text{Pin}_{\mathbb{F}}(V)$. By the above proposition, there is a short exact sequence

$$1 \longrightarrow N_{\mathbb{F}} \longrightarrow \text{Spin}_{\mathbb{F}}(V) \longrightarrow \text{SO}(V) \longrightarrow 1$$

Theorem 3.3.5. *There are isomorphisms $\text{Spin}_{\mathbb{F}}(V) \cong (\text{Spin}(V) \times N_{\mathbb{F}})/(\mathbb{Z}/2\mathbb{Z})$, where $\mathbb{Z}/2\mathbb{Z} \subset \text{Spin}(V) \times N_{\mathbb{F}}$ is identified as the subgroup whose non-identity element is $(-1, -1)$.*

Proof. There is a map $\phi : \text{Spin}(V) \times N_{\mathbb{F}} \rightarrow \text{Spin}_{\mathbb{F}}(V)$ given by $\phi(x, a) = xa$. Clearly $(-1, -1)$ is in the kernel of this map, and it is easy to see that this must be the entire kernel by examining the Clifford algebras. So we have an injective map $(\text{Spin}(V) \times N_{\mathbb{F}})/(\mathbb{Z}/2\mathbb{Z}) \rightarrow \text{Spin}_{\mathbb{F}}(V)$. To see that it is surjective, suppose $x \in \text{Spin}_{\mathbb{F}}(V)$. Let $T \in \text{SO}(V)$ be the its image under the map $\text{Spin}_{\mathbb{F}}(V) \rightarrow \text{SO}(V)$. Then there is a $y \in \text{Spin}(V)$ that maps to T as well, and if we view $y \in \text{Spin}(V)$ as an element of $\text{Spin}_{\mathbb{F}}(V)$, then by the short exact sequence above, it follows that $x = ya$ for some $a \in N_{\mathbb{F}}$. So x is in the image of the map $(\text{Spin}(V) \times N_{\mathbb{F}})/(\mathbb{Z}/2\mathbb{Z}) \rightarrow \text{Spin}_{\mathbb{F}}(V)$, and therefore we have an isomorphism. \square

Theorem 3.3.6. *The map $\text{Spin}(V) \rightarrow \text{SO}(V)$ is a double cover. For any \mathbb{F} and V , $\text{Spin}_{\mathbb{F}}(V)$ is a compact Lie group, and if $\dim(V) \geq 2$, then $\text{Spin}_{\mathbb{F}}(V)$ is a connected Lie group.*

Proof. The short exact sequence implies that $\text{Spin}(V) \rightarrow \text{SO}(V)$ is a double cover. Since $\text{SO}(V)$ is a compact Lie group, this implies $\text{Spin}(V)$ is too, and since $N_{\mathbb{F}}$ is compact, the isomorphisms above imply that $\text{Spin}_{\mathbb{F}}(V)$ is as well.

So we just need to show that $\text{Spin}_{\mathbb{F}}(V)$ is connected, provided that $\dim(V) \geq 2$. And since $N_{\mathbb{F}}$ is connected if $\mathbb{F} \neq \mathbb{R}$, we just need to show that $\text{Spin}(V)$ is connected. And to do this, since the kernel of the map $\text{Spin}(V) \rightarrow \text{SO}(V)$ is $\{\pm 1\}$ and $\text{SO}(V)$ is connected, we can just show that there exists a path connecting 1 to -1 in $\text{Spin}(V)$. Let us assume that Q is negative definite; the positive definite case is similar. So, let e_1, \dots, e_n be an orthonormal basis for V and set

$$\gamma(t) := \cos(\pi t) + e_1 e_2 \sin(\pi t)$$

for $t \in [0, 1]$. Clearly this is a path in $\text{Cl}(V)$ from 1 to -1 , so we just need to show it lies in $\text{Spin}(V)$. The element $\gamma(t)$ has inverse $\cos(\pi t) - e_1 e_2 \sin(\pi t)$, and if $v = a_1 e_1 + a_2 e_2 + w$, where w is in the orthogonal complement of the subspace of V generated by e_1 and e_2 , we have $\alpha(\gamma(t))v\gamma(t)^{-1} \in V$. One way to see this is to note that $\gamma(t) \in \text{Cl}(V)^0$, and when $n = 2$, $\alpha(\gamma(t))v\gamma(t)^{-1} \in \text{Cl}(V)^1 = V$. When $n \geq 2$, this still applies for the $a_1 e_1 + a_2 e_2$ part of v , and $\alpha(\gamma(t)) = \gamma(t)$ simply commutes past w . Since $\alpha(\gamma(t)) = \gamma(t)$ and $\gamma(t)^t = \gamma(t)^{-1}$, we see $N(\gamma(t)) = 1$. Finally, to see that $\gamma(t)$ acts by an element of $\text{SO}(V)$, we only need to worry about e_1 and e_2 , since $\gamma(t)$ acts by the identity on e_3, \dots, e_n . A simple computation reveals that

$$\alpha(\gamma(t))e_1\gamma(t)^{-1} = (\cos^2(\pi t) - \sin^2(\pi t))e_1 + 2\sin(\pi t)\cos(\pi t)e_2,$$

$$\alpha(\gamma(t))e_2\gamma(t)^{-1} = -2\sin(\pi t)\cos(\pi t)e_1 + (\cos^2(\pi t) - \sin^2(\pi t))e_2.$$

In matrix form, this is

$$\begin{bmatrix} \cos^2(\pi t) - \sin^2(\pi t) & -2\sin(\pi t)\cos(\pi t) \\ 2\sin(\pi t)\cos(\pi t) & \cos^2(\pi t) - \sin^2(\pi t) \end{bmatrix}.$$

It is then easy to see that this is in $\text{SO}(\mathbb{R}^2)$. \square

Corollary 3.3.7. *If the dimension of V is at least 2, $\text{Spin}(V)$ is the universal cover of $\text{SO}(V)$.*

One important property of $\text{Spin}_{\mathbb{F}}(V)$ is its relation to the grading.

Proposition 3.3.8. *For any V , $\text{Spin}_{\mathbb{F}}(V) \subset \text{Cl}_{\mathbb{F}}(V)^0$. The non-identity component of $\text{Pin}_{\mathbb{F}}(V)$ belongs to $\text{Cl}_{\mathbb{F}}(V)^1$ if $\dim(V) \geq 2$.*

Proof. Suppose $x \in \text{Pin}_{\mathbb{F}}(V)$. Then its image in $\text{O}(V)$ is a composition of reflections, so we can write this as $v_1 \dots v_n a$ for $v_1, \dots, v_n \in V$ nonzero and $a \in N_{\mathbb{F}}$, where n is the number of reflections. Clearly the image is in $\text{SO}(V)$ if and only if n is even. \square

If $\dim(V) = 1$, then $\text{Pin}(V)$ has three non-identity components. One belongs to $\text{Spin}(V)$, while the other two do not.

Now we discuss the relationship between these groups. A map $V \rightarrow W$ preserving the quadratic form induces a map $\text{Spin}_{\mathbb{F}}(V) \rightarrow \text{Spin}_{\mathbb{F}}(W)$ coming from the map of Clifford algebras. Similarly, if $\mathbb{F}_1 \subseteq \mathbb{F}_2$, then we get a map $\text{Spin}_{\mathbb{F}_1}(V) \rightarrow \text{Spin}_{\mathbb{F}_2}(V)$ again induced by the map of Clifford algebras.

Now as we did before for Clifford algebras, we fix a model for Spin groups in each dimension by setting $\text{Spin}_{\mathbb{F}}(n) = \text{Spin}_{\mathbb{F}}(\mathbb{R}^n)$, where \mathbb{R}^n has the quadratic form $v \mapsto -v \cdot v$. So the maps $\mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$ induce maps $\text{Spin}_{\mathbb{F}}(n) \rightarrow \text{Spin}_{\mathbb{F}}(n+1)$.

The tensor product of Clifford algebras/direct sum of vector spaces also induces pairings of Spin groups. Let $(\mathbb{F}_1, \mathbb{F}_2, \mathbb{F})$ be a compatible triple.

Proposition 3.3.9. *There is a group homomorphism $\text{Spin}_{\mathbb{F}_1}(n_1) \times \text{Spin}_{\mathbb{F}_2}(n_2) \rightarrow \text{Spin}_{\mathbb{F}}(n_1 + n_2)$ given by $(x_1, x_2) \mapsto x_1 x_2$ (using the standard isomorphism $\text{Cl}_{\mathbb{F}_1}(n_1) \otimes_{\mathbb{Z}/2\mathbb{Z}} \text{Spin}_{\mathbb{F}_2}(n_2) \cong \text{Spin}_{\mathbb{F}}(n_1 + n_2)$). This map is such that the diagram*

$$\begin{array}{ccc} \text{Spin}_{\mathbb{F}_1}(n_1) \times \text{Spin}_{\mathbb{F}_2}(n_2) & \longrightarrow & \text{Spin}_{\mathbb{F}}(n_1 + n_2) \\ \downarrow & & \downarrow \\ \text{SO}(n_1) \times \text{SO}(n_2) & \longrightarrow & \text{SO}(n_1 + n_2) \end{array}$$

commutes.

Here the bottom arrow is induced by our standard identification $\mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2} \cong \mathbb{R}^{n_1+n_2}$

Proof. To see the map $(x_1, x_2) \mapsto x_1x_2$ is a group homomorphism, first notice that it is multiplicative at the level of Clifford algebras since all the relevant groups sit inside the even-graded piece of their Clifford algebra, so $x_1x_2y_1y_2 = (x_1y_1)(x_2y_2)$. So we just need to check that $x_1x_2 \in \text{Spin}_{\mathbb{F}}(n_1 + n_2)$. First, if $v = v_1 + v_2 \in \mathbb{R}^{n_1+n_2}$, we have

$$\begin{aligned} x_1x_2v(x_1x_2)^{-1} &= x_1x_2(v_1 + v_2)x_2^{-1}x_1^{-1} \\ &= x_1v_1x_2x_2^{-1}x_1^{-1} + x_2v_2x_2^{-1}x_1x_1^{-1} \\ &= x_1v_1x_1^{-1} + x_2v_2x_2^{-1} \in \mathbb{R}^{n_1+n_2}. \end{aligned}$$

So x_1x_2 is in the Clifford group. We calculate the norm as

$$\begin{aligned} N(x_1x_2) &= x_1x_2\alpha((x_1x_2)^t) \\ &= x_1x_2\alpha(x_2^tx_1^t) \\ &= x_1x_2x_2^tx_1^t \\ &= x_1N(x_2)x_1^t \\ &= N(x_1)N(x_2) \\ &= 1. \end{aligned}$$

Finally, since x_1x_2 acts on $v = v_1 + v_2$ by taking it to $x_1v_1x_1^{-1} + x_2v_2x_2^{-1}$, we see that it acts on $\mathbb{R}^{n_1+n_2}$ as the direct sum of the action of x_1 on \mathbb{R}^{n_1} and the action of x_2 on \mathbb{R}^{n_2} , so $x_1x_2 \in \text{Spin}_{\mathbb{F}}(n_1 + n_2)$ and the diagram commutes. \square

These maps satisfy the same associativity law that the isomorphisms of tensor products of Clifford algebras satisfy. There are also suitable compatibility maps for exchanging the two factors compatible with the maps for the special orthogonal groups.

Finally, it will be important to define “stable groups”, which are infinite-dimensional colimits of $\text{Spin}_{\mathbb{F}}(n)$. We let $\text{Spin}_{\mathbb{F}}$ (along with the inclusions $\text{Spin}_{\mathbb{F}}(n) \rightarrow \text{Spin}_{\mathbb{F}}$) be the colimit of the diagram

$$\text{Spin}_{\mathbb{F}}(0) \longrightarrow \text{Spin}_{\mathbb{F}}(1) \longrightarrow \text{Spin}_{\mathbb{F}}(2) \longrightarrow \text{Spin}_{\mathbb{F}}(3) \longrightarrow \dots$$

in the category of topological spaces. The space $\text{Spin}_{\mathbb{F}}$ has the union topology of all the $\text{Spin}_{\mathbb{F}}(n)$, and therefore the group structures on $\text{Spin}_{\mathbb{F}}(n)$ induce a topological group structure on $\text{Spin}_{\mathbb{F}}$ in a unique way such that the inclusions $\text{Spin}_{\mathbb{F}}(n) \rightarrow \text{Spin}_{\mathbb{F}}$ are homomorphisms.

These groups satisfy properties analogous to their finite counterparts. There are maps $\text{Spin}_{\mathbb{R}} \rightarrow \text{Spin}_{\mathbb{C}}$ and $\text{Spin}_{\mathbb{C}} \rightarrow \text{Spin}_{\mathbb{H}}$, as well as maps $\text{Spin}_{\mathbb{F}} \rightarrow \text{SO}$ (where SO is formed from the $\text{SO}(n)$ in the same way) sitting in exact sequences

$$1 \longrightarrow N_{\mathbb{F}} \longrightarrow \text{Spin}_{\mathbb{F}} \longrightarrow \text{SO} \longrightarrow 1$$

There are also isomorphisms

$$\mathrm{Spin}_{\mathbb{F}} \cong (\mathrm{Spin}_{\mathbb{R}} \times U_k) / (\mathbb{Z}/2\mathbb{Z}).$$

These all follow from the finite-dimensional isomorphisms and their compatibility with the inclusions $\mathrm{Spin}_{\mathbb{F}}(n) \rightarrow \mathrm{Spin}_{\mathbb{F}}(n+1)$.

Chapter 4

The Atiyah–Bott–Shapiro Orientation

The Atiyah–Bott–Shapiro orientation is a fundamental relationship between Spin bordism and K-theory. Its construction, however, does not make a mention of bordism. Instead the construction is geometric and involves Thom spaces, and the relationship with bordism enters through the Pontryagin–Thom isomorphism. Our goal in this section is to define and explain basic properties of this relationship.

4.1 K-Theory

As we know from homotopy theory, a generalized cohomology can be defined as a contravariant functor from the homotopy category of spaces to the category of graded abelian groups that satisfies some specific axioms. It is well known from the theory of vector bundles that the pullbacks of a vector bundle along homotopic maps are isomorphic. Moreover, we have the operation of *Whitney sum* on vector bundles, which is associative and commutative and has an identity up to coherent isomorphisms. So isomorphism classes of vector bundles and pulling them back gets us close to such a functor, but the problem is that we do not have inverses under the Whitney sum. The *Grothendieck construction* remedies this by formally introducing inverses.

Definition 4.1.1. Let X be a set equipped with a binary operation $+$ that is commutative and associative, i.e. a semigroup. Then its **group completion** or **Grothendieck group** is the abelian group $K(X)$, defined as the free abelian group generated by the elements of X quotiented by the all the relations $[x] + [y] - [x + y]$ for $x, y \in X$ (here $[x]$ denotes the generator of the free group associated to x).

The important thing about $K(X)$ is not the specific way it is constructed, but rather its universal property. Note that there is a canonical map $i : X \rightarrow K(X)$ taking $x \in X$ to $[x]$, and the relations in $K(X)$ ensure that this map is a homomorphism ($f(x + y) = f(x) + f(y)$ for all $x, y \in X$).

Proposition 4.1.2. *Suppose that A is an abelian group and $f : X \rightarrow A$ is a homomorphism. Then there is a unique homomorphism $\tilde{f} : K(X) \rightarrow A$ such that $f = \tilde{f}i$.*

In particular, a homomorphism $X \rightarrow Y$ induces a homomorphism of groups $K(X) \rightarrow K(Y)$; this is functorial.

In the presence of multiplicative structure on X , $K(X)$ inherits a compatible multiplication. It will be convenient for us to state this in full generality.

Proposition 4.1.3. *If X , Y , and Z are semigroups and $m : X \times Y \rightarrow Z$ is a bilinear map, then there is a unique map $m : K(X) \otimes K(Y) \rightarrow K(Z)$ such that $m([x] \otimes [y]) = [m(x, y)]$*

Proof. For each $x \in X$, the map $y \mapsto m(x, y)$ is a homomorphism of semigroups $Y \rightarrow Z$ by bilinearity, and thus determines a map $K(Y) \rightarrow K(Z)$ with $[y] \mapsto [m(x, y)]$. Then the map taking $x \in X$ to the left multiplication map $K(Y) \rightarrow K(Z)$ is also a homomorphism by bilinearity, and so we get a homomorphism $K(X) \rightarrow \text{Hom}(K(Y), K(Z))$. This is adjoint to $m : K(X) \otimes K(Y) \rightarrow K(Z)$, which takes $[x] \otimes [y]$ to $[m(x, y)]$. Uniqueness is clear. \square

If $X = Y = Z$ is a semiring, then this makes $K(X)$ into a ring.

We will now discuss K-theory in the context of (graded) Clifford modules, and then return to the motivating example of topological K-theory.

4.1.1 The K-Theory of Clifford Modules

As mentioned above in our study of Clifford algebras and their modules, K-theory will provide the best setting for us to understand the way in which Clifford modules over different Clifford algebras are related. So, let $M_{\mathbb{F}}(n)$ be the Grothendieck group of isomorphism classes of finitely-generated graded left $\text{Cl}_{\mathbb{F}}(n)$ -modules under direct sum. Now, there are restriction homomorphisms $i_{\mathbb{F},n} : M_{\mathbb{F}}(n+1) \rightarrow M_{\mathbb{F}}(n)$ induced by the inclusion $\text{Cl}_{\mathbb{F}}(n) \rightarrow \text{Cl}_{\mathbb{F}}(n+1)$. Define $A_{\mathbb{F}}(n)$ to be the cokernel of this map. Using Theorem 3.2.4 and the fact that graded left $\text{Cl}_{\mathbb{F}}(n+1)$ -modules are equivalent to left $\text{Cl}_{\mathbb{F}}(n)$ -modules, we can describe the groups $A_{\mathbb{F}}(n)$:

Theorem 4.1.4. *The groups $A_{\mathbb{F}}(n)$ are isomorphic to the following groups:*

n	$A_{\mathbb{R}}(n)$	$A_{\mathbb{C}}(n)$	$A_{\mathbb{H}}(n)$
0	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}
1	$\mathbb{Z}/2\mathbb{Z}$	0	0
2	$\mathbb{Z}/2\mathbb{Z}$	\mathbb{Z}	0
3	0	0	0
4	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}
5	0	0	$\mathbb{Z}/2\mathbb{Z}$
6	0	\mathbb{Z}	$\mathbb{Z}/2\mathbb{Z}$
7	0	0	0
8	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}

And there are isomorphisms

$$\begin{aligned} A_{\mathbb{R}}(n+8) &\cong A_{\mathbb{R}}(n), \\ A_{\mathbb{C}}(n+2) &\cong A_{\mathbb{C}}(n), \\ A_{\mathbb{H}}(n+8) &\cong A_{\mathbb{H}}(n), \\ A_{\mathbb{H}}(n+4) &\cong A_{\mathbb{R}}(n). \end{aligned}$$

Proof. Since the finitely-generated modules over $\mathbb{F}_0(n)$ are all direct sums of \mathbb{F}_0^n , we can easily determine the \mathbb{R} -dimension of the even-graded part of each generator of $M_{\mathbb{F}}(n+1)$ by finding the product of matrix algebras that $\text{Cl}_{\mathbb{F}}(n)$ is isomorphic to. Then by comparing dimensions, it is easy to see what $i_{\mathbb{F},n}$ does to the generators of $M_{\mathbb{F}}(n+1)$ except when $\mathbb{F} = \mathbb{R}, \mathbb{H}$ and $n = 4, 8, \dots$ or $\mathbb{F} = \mathbb{C}$ and $n = 2, 4, 6, 8, \dots$

To fill in these missing entries in the table, we can use the grading involution $(-)_*$. Since this involution commutes with direct sums, it induces automorphisms $(-)_* : M_{\mathbb{F}}(n) \rightarrow M_{\mathbb{F}}(n)$. Moreover, these automorphisms clearly commute with the restriction maps $i_{\mathbb{F},n}$. Now, since there is only one irreducible graded $\text{Cl}_{\mathbb{F}}(n)$ -module for $\mathbb{F} = \mathbb{R}, \mathbb{H}$ and $n \neq 4, 8 \pmod{8}$ and for $\mathbb{F} = \mathbb{C}$ and $n \neq 2 \pmod{2}$, the grading involution takes this generator to itself, and therefore it acts by the identity on $M_{\mathbb{F}}(n)$ for these choices of \mathbb{F} and n . And then as we saw in Proposition 3.2.8, when $M_{\mathbb{F}}(n) \cong \mathbb{Z} \oplus \mathbb{Z}$, $(-)_*$ exchanges these two summands. So if the codomain of $i_{\mathbb{F},n}$ is \mathbb{Z}^2 and the domain is \mathbb{Z} , then $A_{\mathbb{F}}(n) \cong \mathbb{Z}$, and if the codomain is \mathbb{Z} and the domain is \mathbb{Z}^2 , then $A_{\mathbb{F}}(n) \cong 0$, by examining the \mathbb{R} -dimension of the irreducible modules. \square

To make full use of these groups, we will need to bring over the multiplicative structure induced by the tensor product. Form (\mathbb{Z}) -graded abelian groups by setting $M_{\mathbb{F}}^* = \bigoplus_{n \geq 0} M_{\mathbb{F}}(n)$ and $A_{\mathbb{F}}^* = \bigoplus_{n > 0} A_{\mathbb{F}}(n)$. Then from our discussion of tensor products of Clifford modules, we see that if $\mathbb{F} = \mathbb{R}, \mathbb{C}$, then $M_{\mathbb{F}}^*$ has a product operation that is associative and distributes over addition. Moreover, the module \mathbb{F} (concentrated in degree zero) over $\text{Cl}_{\mathbb{F}}(0) \cong \mathbb{F}$ is the unit for the $\mathbb{Z}/2\mathbb{Z}$ -graded tensor product of \mathbb{F} -vector spaces, and so $M_{\mathbb{F}}^*$ is a graded ring. In the case that $\mathbb{F} = \mathbb{H}$, we have an action of $M_{\mathbb{R}}^*$ that is associative and distributive, and therefore $M_{\mathbb{H}}^*$ has the structure of a graded module over $M_{\mathbb{R}}^*$.

This multiplicative structure carries over to $A_{\mathbb{F}}^*$. To see this in the case \mathbb{F} is commutative, we need to show that the image of all the $i_{\mathbb{F},n}$ is an ideal. So, suppose $a_1 \in M_{\mathbb{F}}^{n_1}$ and $a_2 \in M_{\mathbb{F}}^{n_2+1}$. We want to show that $a_1 i_{\mathbb{F},n_2}(a_2) = i_{\mathbb{F},n_1+n_2}(a_1 a_2)$. It will suffice to do this for a_1 and a_2 represented by graded Clifford modules M_1 and M_2 over $\text{Cl}_{\mathbb{F}}(n_1)$ and $\text{Cl}_{\mathbb{F}}(n_2+1)$, respectively. The graded \mathbb{F} -vector space structure on $a_1 i_{\mathbb{F},n_2}(a_2)$ and $i_{\mathbb{F},n_1+n_2}(a_1 a_2)$ is identical, so we just need to compare the actions of $\text{Cl}_{\mathbb{F}}(n_1+n_2)$. If $i \leq n_1$, then $e_i \in \text{Cl}_{\mathbb{F}}(n_1+n_2)$ acts by $e_i \otimes 1$ for both, and if $n_1 < i \leq n_1+n_2$, then e_i acts by $1 \otimes e_{i-n_1}$ with the same sign rule for both modules. Hence $a_1 i_{\mathbb{F},n_2}(a_2) = i_{\mathbb{F},n_1+n_2}(a_1 a_2)$, and therefore the subgroup generated by the image of all the $i_{\mathbb{F},n}$ is an ideal. So the ring structure on $M_{\mathbb{F}}^*$ induces ring structure on $A_{\mathbb{F}}^*$.

In the case that $\mathbb{F} = \mathbb{H}$, we need to show that the image of all the $i_{\mathbb{H},n}$ is a submodule, and that this submodule is closed under the action of the image of $i_{\mathbb{R},n}$. This is true for the exact same reason as above. So $A_{\mathbb{H}}^*$ has the structure of a module over $A_{\mathbb{R}}^*$.

Theorem 4.1.5. *There are isomorphisms of graded rings*

$$\begin{aligned} A_{\mathbb{R}} &\cong \mathbb{Z}[\xi_1, \mu_4, \lambda_8]/(2\xi_1, \xi_1^3, \mu_4^2 - 4\lambda_8), \\ A_{\mathbb{C}} &\cong \mathbb{Z}[\mu_2]. \end{aligned}$$

As a module over $A_{\mathbb{R}}$, $A_{\mathbb{H}}$ is isomorphic to $A_{\mathbb{R}}$ but shifted down by four degrees and truncated to be zero below degree zero.

Note that the degrees of the generators are denoted by the subscripts. For $A_{\mathbb{H}}$, we let the generator in degree zero be denoted μ_0 , while the generators of degrees four and five are denote λ_4 and ξ_5 .

Proof. For $A_{\mathbb{R}}$, note that multiplication by a generator of $A_{\mathbb{R}}(8)$ induces an isomorphism of abelian groups, since the the dimension of the even part of an irreducible graded $\text{Cl}(8)$ -module is 8, and the isomorphism $\text{Cl}(n+8) \cong \text{Cl}(n) \otimes_{\mathbb{Z}/2\mathbb{Z}} \text{Cl}(8) \cong \text{Cl}(n) \otimes_{\mathbb{Z}/2\mathbb{Z}} \mathbb{R}(16)$ ensures that the dimension of the even pieces of irreducible graded $\text{Cl}(n+8)$ -modules are 16 times the dimension of the even-graded parts of irreducible graded $\text{Cl}(n)$ -modules. If ξ_1 is the generator of $A_{\mathbb{R}}(1) \cong \mathbb{Z}/2\mathbb{Z}$, then clearly $2\xi_1 = 0$ and $\xi_1^3 = 0$. To see that $\xi_1^2 \neq 0$, we only need to see that the dimension of ξ_1 as an irreducible graded $\text{Cl}(1)$ -module is two, and the dimension of an irreducible graded $\text{Cl}(2)$ -module is four. Finally, letting μ_4 be the generator of $A_{\mathbb{R}}(4)$, represented by an irreducible graded $\text{Cl}(4)$ -module, we see that this has dimension eight, and therefore its square has dimension 64, so it is $ax_1 + bx_2$, where $a + b = 4$, $a, b \geq 0$, and x_1 and x_2 are the generators for the two summands of $M_{\mathbb{R}}(8)$.

To see that either $a = 0$ or $b = 0$, we must return to the Clifford algebras. Recall that $\text{Cl}(4n-1)$ is always isomorphic a direct sum of two identical matrix algebras, and that the summands can be identified as $(1 \pm \omega)\text{Cl}(4n-1)$, where $\omega = e_1e_2 \dots e_{4n-1}$. Using the isomorphism $\text{Cl}(4n-1) \cong \text{Cl}(4n)^0$, we see that ω corresponds to $\omega_{4n} = e_1e_2 \dots e_{4n-1}e_{4n}$. So ω_{4n} acts on each irreducible $\text{Cl}(4n)^0$ -module as ± 1 . When we form the tensor product $\text{Cl}(4n) \otimes_{\mathbb{Z}/2\mathbb{Z}} \text{Cl}(4n) \cong \text{Cl}(8n)$, it is easy to see that $\omega_{4n} \otimes \omega_{4n}$ corresponds to ω_{8n} under the usual isomorphism. So the tensor product of two irreducible graded $\text{Cl}(4n)$ -modules where ω_{4n} acts by $+1$ on the zero-graded piece is such that ω_{8n} acts by $+1$ on the zero-graded piece. In particular, we see that $a = 0$ or $b = 0$. Then we can choose our generator $\lambda_8 \in A_{\mathbb{R}}(8)$ such that $\mu_4^2 = 4\lambda_8$.

Using the same dimension counting and grading involution arguments, it is clear that when $\mathbb{F} = \mathbb{C}$, multiplication by a generator of $A_{\mathbb{C}}(2)$ is an isomorphism, and hence we can immediately deduce the result.

For $\mathbb{F} = \mathbb{H}$, the same arguments as the real case apply. □

4.1.2 Topological K-Theory and The Difference Bundle

Now, we return to the motivating example of K-theory, the K-theory of a topological space. We define this by forming the group completion of the set of isomorphism classes of vector bundles under direct sum:

Definition 4.1.6. If X is a compact CW complex, then $K\mathbb{F}^{-n}(X)$ for $n \geq 0$ is the Grothendieck group $K(Y_n)$, where Y_n is the set of isomorphism classes of \mathbb{F} -vector bundles on $\Sigma^n X$ equipped with the operation of Whitney sum.

Then $K\mathbb{F}^{-n}$ is a contravariant functor from the homotopy category of compact CW-complexes to abelian groups. For compact CW-complexes, there are long exact sequences to the left for cofiber sequences (see [Ati67]). So we are close to a cohomology theory, but it is only defined in nonpositive degrees and for compact CW complexes. To get an actual cohomology theory, we use the following results:

Theorem 4.1.7. *The functor $\tilde{K}\mathbb{F}^0$ is represented by the (pointed) space $\mathbb{Z} \times BGL_{\mathbb{F}}$, i.e. there is a natural isomorphism*

$$\tilde{K}\mathbb{F}^0(-) \cong [-, \mathbb{Z} \times BGL_{\mathbb{F}}]_*.$$

Theorem 4.1.8 (Bott Periodicity). *There are the following equivalences:*

$$\begin{aligned} \Omega^8 BGL_{\mathbb{R}} &\cong \mathbb{Z} \times BGL_{\mathbb{R}}, \\ \Omega^2 BGL_{\mathbb{C}} &\cong \mathbb{Z} \times BGL_{\mathbb{C}}, \\ \Omega^8 BGL_{\mathbb{H}} &\cong \mathbb{Z} \times BGL_{\mathbb{H}}, \\ \Omega^4 BGL_{\mathbb{H}} &\cong \mathbb{Z} \times BGL_{\mathbb{R}}. \end{aligned}$$

Note that these imply $\Omega^4 BGL_{\mathbb{R}} \cong \mathbb{Z} \times BGL_{\mathbb{H}}$. So, we can now define K-theory in all degrees and for all spaces by letting

$$K\mathbb{F}^n(X) := [X, \Omega^{k_n} BGL_{\mathbb{F}}],$$

where $k_n \in \{1, \dots, p_{\mathbb{F}}\}$ is equal to $-n$ modulo $p_{\mathbb{F}}$, where $p_{\mathbb{F}}$ is the *period* of $BGL_{\mathbb{F}}$ (i.e. $p_{\mathbb{R}} = p_{\mathbb{H}} = 8$, $p_{\mathbb{C}} = 2$). This is **representable K-theory**, and it is a generalized cohomology theory. For compact CW complexes, and nonpositive degrees, it is equivalent to what we defined above, but for different kinds of spaces $K\mathbb{F}^0$ need not be the Grothendieck group of vector bundles.

These cohomology theories can be represented by spectra $K\mathbb{F}$ whose n th space is $\Omega^{k_n} BGL_{\mathbb{F}}$. At the level of spectra, Bott periodicity manifests as equivalences

$$\Omega^8 K\mathbb{R} \cong K\mathbb{R},$$

$$\begin{aligned}\Omega^2 K\mathbb{C} &\cong K\mathbb{C}, \\ \Omega^8 K\mathbb{H} &\cong K\mathbb{H}, \\ \Omega^4 K\mathbb{H} &\cong K\mathbb{R}.\end{aligned}$$

Now, the spectra $K\mathbb{R}$ and $K\mathbb{C}$ admit extra structure as commutative ring spectra. These are induced by the tensor product of vector bundles, which is encoded by maps $BGL_{\mathbb{F}} \wedge BGL_{\mathbb{F}} \rightarrow BGL_{\mathbb{F}}$. For \mathbb{H} , we cannot get a quaternionic vector space by tensoring two quaternionic vector spaces, so there is no analagous ring structure. However, we can tensor a real vector space and a quaternionic vector space to get another quaternionic vector spaces, so adapting this to vector bundles, we get a map $BGL_{\mathbb{R}} \wedge BGL_{\mathbb{H}} \rightarrow BGL_{\mathbb{H}}$. And at the spectrum level, this gives $K\mathbb{H}$ the structure of a $K\mathbb{R}$ -module.

So in particular, the homotopy groups $\pi_* K\mathbb{F}$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$ admit the structure of graded-commutative rings, while $\pi_* K\mathbb{H}$ has the structure of a module over $\pi_* K\mathbb{R}$.

Theorem 4.1.9. *The homotopy groups $\pi_* K\mathbb{F}$ have the following ring/module structures:*

$$\begin{aligned}\pi_* K\mathbb{R} &\cong \mathbb{Z}[\eta_1, x_4, v_8, v_8^{-1}] / (2\eta_1, \eta_1^3, \eta_1 x_4, x_4^2 - 4v_8), \\ \pi_* K\mathbb{C} &\cong \mathbb{Z}[x_2, x_2^{-1}].\end{aligned}$$

As a module over $K\mathbb{R}^$, $K\mathbb{H}^*$ is isomorphic to $K\mathbb{R}^*$ but with a shift by four degrees.*

So for $\pi_* K\mathbb{H}$, we have generators x_0, v_4 , and η_1 .

Proof. See [Kar78, Theorem III.5.19]. □

For complex K-theory, the class $x_2 \in K\mathbb{C}^{-2}(\ast) \cong \tilde{K}\mathbb{C}^0(S^2)$ is given by the reduced canonical bundle over $\mathbb{C}\mathbb{P}^1 \cong S^2$. For real K-theory, the generators of this ring have the following description: $\eta_1 \in K\mathbb{R}^{-1}(\ast) \cong K\mathbb{R}^0(S^1)$ is given by the reduction of the canonical bundle over $\mathbb{R}\mathbb{P}^1 \cong S^1$. The class v_8 goes to x_2^4 under complexification. The Bott periodicity equivalences are induced by multiplication with v_8 for \mathbb{R} and \mathbb{H} and multiplication by x_2 for \mathbb{C} .

In order to produce the K-theory classes necessary to build the orientation, we will need a special construction, known as the **difference bundle** construction. Suppose that we have a CW complex X and a subcomplex $Y \subseteq X$. Then we can form a space $A = X \coprod_Y X$ with subspaces $X_0, X_1 \subseteq A$ homeomorphic to X given by each of the summands. There are retractions $\pi_i : A \rightarrow X_i$ given by the identity on X_i and given by folding the other copy of X over onto X_i . Then the long exact sequence of a pair gives us an exact sequence

$$K\mathbb{F}^{n-1}(X_i) \longrightarrow K\mathbb{F}^n(A, X_i) \longrightarrow K\mathbb{F}^n(A) \longrightarrow K\mathbb{F}^n(X_i)$$

But the arrow on the right is a split surjection because of the retraction π_i , and therefore we have split short exact sequences

$$0 \longrightarrow K\mathbb{F}^n(A, X_i) \xrightarrow{\rho_i} K\mathbb{F}^n(A) \xrightarrow{\iota_i} K\mathbb{F}^n(X_i) \longrightarrow 0$$

We know that the map $K\mathbb{F}^n(A, X_i) \rightarrow K\mathbb{F}^n(A)$ is injective by the surjectivity of $K\mathbb{F}^{n-1}(A) \rightarrow K\mathbb{F}^{n-1}(X_i)$ and the long exact sequence. By excision, the map $\phi_i : K\mathbb{F}^n(A, X_i) \rightarrow K\mathbb{F}^n(X_{i+1}, Y)$ induced by the inclusion $(X_{i+1}, Y) \rightarrow (A, X_i)$ is an isomorphism.

Let $\mathcal{C}^1(X, Y)$ denote the collection of tuples $\xi = (\xi_1, \xi_0, \sigma)$, where the ξ_i are vector bundles on X and $\sigma : \xi_1|_Y \rightarrow \xi_0|_Y$ is an isomorphism of vector bundles on Y . Then for $\xi \in \mathcal{C}^1(X, Y)$, we can define a vector bundle $\tilde{\xi}$ on A such that $\tilde{\xi}|_{X_i} \cong \xi_i$ and that the isomorphism $\xi_1|_A \rightarrow \xi_0|_A$ obtained by combining these isomorphisms is σ .

Definition 4.1.10. The **difference bundle** $\chi(\xi) \in K\mathbb{F}^0(X, Y)$ is defined to be the unique class satisfying the equation

$$\rho_1 \phi_0^{-1} \chi(\xi) = \tilde{\xi} - \pi_1^* \xi_1.$$

Note that χ is additive in ξ . That is, if we have $\xi_{i,j}$ and $\sigma_j : \xi_{1,j}|_Y \rightarrow \xi_{0,j}|_Y$ isomorphisms, then we can let $\xi_i = \bigoplus_j \xi_{i,j}$ and $\sigma : \xi_0|_Y \rightarrow \xi_1|_Y$ be $\sigma = \bigoplus_j \sigma_j$, then $\tilde{\xi} \cong \bigoplus_j \tilde{\xi}_j$ as bundles on A , and so

$$\chi(\xi) = \sum_j \chi(\xi_j),$$

where $\xi = (\xi_1, \xi_0, \sigma)$ and $\xi_j = (\xi_{1,j}, \xi_{0,j}, \sigma_j)$. This construction is also natural under pullbacks along maps of pairs $(X', Y') \rightarrow (X, Y)$.

This construction also has nice properties when it comes to tensor products, but it a little more subtle.

Proposition 4.1.11. *Suppose $\xi = (\xi_1, \xi_0, \sigma)$ and $\xi' = (\xi'_1, \xi'_0, \sigma')$ are in $\mathcal{C}(X, Y)$, and that all of these bundles have metrics. Define $\xi'' = (\xi''_1, \xi''_0, \sigma'')$ by setting*

$$\begin{aligned} \xi''_1 &:= (\xi_0 \otimes \xi'_1) \oplus (\xi_1 \otimes \xi'_0), \\ \xi''_0 &:= (\xi_0 \otimes \xi'_0) \oplus (\xi_1 \otimes \xi'_1), \\ \sigma'' &:= \begin{bmatrix} \text{id} \otimes \sigma' & \sigma \otimes \text{id} \\ \sigma^* \otimes \text{id} & -\text{id} \otimes \sigma'^* \end{bmatrix}. \end{aligned}$$

Then $\chi(\xi'') = \chi(\xi)\chi(\xi')$.

The columns of the matrix above are $\xi_0 \otimes \xi'_0$ and $\xi_1 \otimes \xi'_1$ and the rows are $\xi_0 \otimes \xi'_1$ and $\xi_1 \otimes \xi'_0$ in that order. The maps σ^* and σ'^* are the adjoint maps to σ and σ' , respectively. Here ξ and ξ' can be over different fields, so long as they have a compatible tensor product, i.e. it is not true that $\mathbb{F} = \mathbb{F}' = \mathbb{H}$.

Proof. First, we should check that ξ'' is well-defined. We need to show that σ'' is an isomorphism. Using elementary row operations (i.e. composing with some isomorphisms), we see that σ'' is similar to the map given by the matrix

$$\begin{bmatrix} \text{id} \otimes \text{id} & \sigma \otimes \sigma'^{-1} \\ 0 & \sigma^{*-1} \otimes \sigma'^* + \sigma \otimes \sigma'^{-1} \end{bmatrix}$$

from $(\xi_0 \otimes \xi'_1) \oplus (\xi_1 \otimes \xi'_0)$ to $(\xi_0 \otimes \xi'_1) \oplus (\xi_1 \otimes \xi'_1)$. So if $T = (\text{id} \otimes \sigma' \sigma'^*) + (\sigma \sigma^* \otimes \text{id})$ is an automorphism of $\xi_0 \otimes \xi'_0$, then σ'' will be an isomorphism. Now, by the spectral theorem (which holds over the quaternions, see [FP03]), since $\sigma \sigma^*$ and $\sigma' \sigma'^*$ are Hermitian, they are therefore diagonalized by some orthonormal bases in each fiber. Moreover, all the eigenvalues are positive real numbers. So T is diagonalizable in each fiber with all positive eigenvalues, and therefore T is invertible in each fiber.

We want to show that

$$\rho_1 \phi_0^{-1}(\chi(\xi)\chi(\xi')) = \tilde{\xi}'' - \pi_1^* \xi_1''.$$

Note that the maps ρ_i , ι_i , and ϕ_i are all multiplicative in the appropriate sense. So

$$\begin{aligned} \rho_1 \phi_0^{-1}(\chi(\xi)\chi(\xi')) &= (\rho_1 \phi_0^{-1} \chi(\xi))(\rho_1 \phi_0^{-1} \chi(\xi')) \\ &= (\tilde{\xi} - \pi_1^* \xi_1)(\tilde{\xi}' - \pi_1^* \xi_1') \\ &= \tilde{\xi} \tilde{\xi}' + (\pi_1^* \xi_1)(\pi_1^* \xi_1') - (\pi_1^* \xi_1) \tilde{\xi}' - \tilde{\xi}(\pi_1^* \xi_1'). \end{aligned}$$

Hence we just need to show the equation

$$\tilde{\xi} \tilde{\xi}' + (\pi_1^* \xi_1)(\pi_1^* \xi_1') + \pi_1^* \xi_1'' = \tilde{\xi}'' + (\pi_1^* \xi_1) \tilde{\xi}' + \tilde{\xi}(\pi_1^* \xi_1')$$

is true. We can do this by exhibiting an isomorphism of vector bundles

$$(\tilde{\xi} \otimes \tilde{\xi}') \oplus (\pi_1^* \xi_1 \otimes \pi_1^* \xi_1') \oplus \pi_1^* \xi_1'' \cong \tilde{\xi}'' \oplus (\pi_1^* \xi_1 \otimes \tilde{\xi}') \oplus (\tilde{\xi} \otimes \pi_1^* \xi_1')$$

on A . To do this, we restrict these bundles to each X_i with an isomorphism to bundles we understand, and then examine the isomorphism of the two bundles when restricted to A . This completely determines these bundles up to isomorphism. We find

$$\begin{aligned} ((\tilde{\xi} \otimes \tilde{\xi}') \oplus (\pi_1^* \xi_1 \otimes \pi_1^* \xi_1') \oplus \pi_1^* \xi_1'')|_{X_0} &\cong (\xi_0 \otimes \xi'_0) \oplus (\xi_1 \otimes \xi'_1) \oplus (\xi_0 \otimes \xi'_1) \oplus (\xi_1 \otimes \xi'_0), \\ ((\tilde{\xi} \otimes \tilde{\xi}') \oplus (\pi_1^* \xi_1 \otimes \pi_1^* \xi_1') \oplus \pi_1^* \xi_1'')|_{X_1} &\cong (\xi_1 \otimes \xi'_1) \oplus (\xi_1 \otimes \xi'_1) \oplus (\xi_0 \otimes \xi'_1) \oplus (\xi_1 \otimes \xi'_0), \\ ((\tilde{\xi}'' \oplus (\pi_1^* \xi_1 \otimes \tilde{\xi}') \oplus (\tilde{\xi} \otimes \pi_1^* \xi_1'))|_{X_0} &\cong (\xi_0 \otimes \xi'_0) \oplus (\xi_1 \otimes \xi'_1) \oplus (\xi_1 \otimes \xi'_0) \oplus (\xi_0 \otimes \xi'_1), \\ ((\tilde{\xi}'' \oplus (\pi_1^* \xi_1 \otimes \tilde{\xi}') \oplus (\tilde{\xi} \otimes \pi_1^* \xi_1'))|_{X_1} &\cong (\xi_0 \otimes \xi'_1) \oplus (\xi_1 \otimes \xi'_0) \oplus (\xi_1 \otimes \xi'_1) \oplus (\xi_1 \otimes \xi'_1). \end{aligned}$$

The isomorphisms from the restrictions to X_1 to the restrictions to X_0 for each bundle on Y are given by the matrices

$$\begin{bmatrix} 0 & 0 & \sigma \otimes \sigma' & 0 \\ 0 & 0 & 0 & \text{id} \\ \text{id} & 0 & 0 & 0 \\ 0 & \text{id} & 0 & 0 \end{bmatrix}$$

for $(\tilde{\xi} \otimes \tilde{\xi}') \oplus (\pi_1^* \xi_1 \otimes \pi_1^* \xi'_1) \oplus \pi_1^* \xi''_1$ and

$$\begin{bmatrix} \text{id} \otimes \sigma' & \sigma \otimes \text{id} & 0 & 0 \\ \sigma^* \otimes \text{id} & -\text{id} \otimes \sigma'^* & 0 & 0 \\ 0 & 0 & \sigma \otimes \text{id} & 0 \\ 0 & 0 & 0 & \text{id} \otimes \sigma' \end{bmatrix}$$

for $(\tilde{\xi}'' \oplus (\pi_1^* \xi_1 \otimes \tilde{\xi}') \oplus (\tilde{\xi} \otimes \pi_1^* \xi'_1))$. Here the columns are $\xi_0 \otimes \xi'_1$, $\xi_1 \otimes \xi'_0$, $\xi_1 \otimes \xi'_1$, $\xi_1 \otimes \xi'_1$ and the rows are $\xi_0 \otimes \xi'_0$, $\xi_1 \otimes \xi'_1$, $\xi_0 \otimes \xi'_1$, $\xi_1 \otimes \xi'_0$ in that order. We then get an isomorphism from $(\tilde{\xi}'' \oplus (\pi_1^* \xi_1 \otimes \tilde{\xi}') \oplus (\tilde{\xi} \otimes \pi_1^* \xi'_1))$ to $(\xi \otimes \xi') \oplus (\pi_1^* \xi_1 \otimes \pi_1^* \xi'_1) \oplus \pi_1^* \xi''_1$ in terms of the restrictions above given by the identity of $(\xi_0 \otimes \xi'_0) \oplus (\xi_1 \otimes \xi'_1) \oplus (\xi_0 \otimes \xi'_1) \oplus (\xi_1 \otimes \xi'_0)$ on X_0 and the matrix

$$\begin{bmatrix} 0 & 0 & \sigma \otimes \text{id} & 0 \\ 0 & 0 & 0 & \text{id} \otimes \sigma' \\ \sigma^{-1} \otimes \text{id} & \text{id} \otimes \sigma'^{-1} & 0 & 0 \\ \sigma^* \otimes \text{id} & -\text{id} \otimes \sigma'^* & 0 & 0 \end{bmatrix}$$

on X_1 . This is a well-defined map because

$$\begin{aligned} & \begin{bmatrix} \text{id} \otimes \sigma' & \sigma \otimes \text{id} & 0 & 0 \\ \sigma^* \otimes \text{id} & -\text{id} \otimes \sigma'^* & 0 & 0 \\ 0 & 0 & \sigma \otimes \text{id} & 0 \\ 0 & 0 & 0 & \text{id} \otimes \sigma' \end{bmatrix} \\ = & \begin{bmatrix} 0 & 0 & \sigma \otimes \sigma' & 0 \\ 0 & 0 & 0 & \text{id} \\ \text{id} & 0 & 0 & 0 \\ 0 & \text{id} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \sigma \otimes \text{id} & 0 \\ 0 & 0 & 0 & \text{id} \otimes \sigma' \\ \sigma^{-1} \otimes \text{id} & \text{id} \otimes \sigma'^{-1} & 0 & 0 \\ \sigma^* \otimes \text{id} & -\text{id} \otimes \sigma'^* & 0 & 0 \end{bmatrix} \end{aligned}$$

This map is invertible as σ'' is invertible. \square

The tensor product rule also takes a similar form for “external” products. If $\xi' = (\xi'_1, \xi'_0, \sigma')$ is instead an element of $\mathcal{C}(X', Y')$, then we can define $\xi'' = (\xi''_1, \xi''_0, \sigma'') \in \mathcal{C}(X \times X', X \times Y' \cup Y \times X')$ as before, but anywhere ξ_i and ξ'_i occur in the “internal” definition, we replace them by the bundles gotten by pulling back along the projections $p : X \times X' \rightarrow X$ and $p' : X \times X' \rightarrow X'$. Then we get the equation

$$\chi(\xi'') = (p^* \chi(\xi))(p'^* \chi(\xi')) = \chi(p^* \xi) \chi(p'^* \xi').$$

Finally, we give a condition for the difference bundle to vanish:

Lemma 4.1.12. *Suppose ξ is such that the isomorphism σ can be extended to all of X . Then $\chi(\xi) = 0$.*

Proof. Choosing an extension ξ_X of ξ to all of X , we see that $\chi(\xi)$ is the pullback of $\chi(\xi_X)$ along $(X, Y) \rightarrow (X, X)$. But $\chi(\xi_X) \in K\mathbb{F}^0(X, X) \cong 0$. \square

In [ABS64], this construction is explored in much greater depth. There, sequences of $n + 1$ vector bundles on X with an exact sequence of maps on Y are considered, forming a category $\mathcal{C}^n(X, Y)$. By imposing an equivalence relation on these sequences analagous to stable isomorphism for vector bundles, one obtains a commutative semigroup. It is shown that these semigroups are isomorphic for all lengths. Then, they consider the notion of an **Euler characteristic**, which is a natural transformation from this semigroup to $K\mathbb{F}(X, Y)$ such that when $Y = \emptyset$, it is the alternating sum of the vector bundles in the sequence. They go on to show that such an Euler characteristic must be an isomorphism, and in the case $n = 1$, the difference bundle is such an Euler characteristic. Finally, a product operation on these sequences is introduced, and it is shown that the Euler characteristics are multiplicative in a compatible way. This is how the multiplicative property of the difference bundle is proven.

However, these results are reliant on X being a finite CW complex, or at least finite-dimensional. For the formulation of the Atiyah–Bott–Shapiro orientation in spectra, we will need everything to be valid over classifying spaces, which are generally not finite-dimensional, and we do not need sequences of greater length for our treatment of the splitting.

4.2 Orientations and Clifford Bundles

The idea of orientation is ubiquitous and has many different incarnations. In homotopy theory, the concept of orientation is generalized and abstracted quite a bit, so that it we can discuss it for any multiplicative generalized cohomology theory.

Definition 4.2.1. Let X be a space with a vector bundle $\xi : E \rightarrow X$ equipped with a metric. The **unit disk bundle** $D(\xi)$ and **unit sphere bundle** $S(\xi)$ of ξ are defined by

$$D(\xi) := \{v \in E \mid |v| \leq 1\}$$

and

$$S(\xi) := \{v \in E \mid |v| = 1\}.$$

Then the **Thom space** of ξ is $T\xi := D(\xi)/S(\xi)$.

Note that the Thom space is naturally equipped with a base point ∞ , which is the point that $S(\xi)$ is collapsed to.

Definition 4.2.2. Let E be a ring spectrum and let ξ be a vector bundle on a space X . Then an **E -orientation** of ξ is an element $\omega \in E^*T\xi$ such that for any $x \in X$, if $i_x : D(\xi)_x/S(\xi)_x \rightarrow T\xi$ is the map induced by the inclusion of the unit disk above x into the unit disk bundle, then $i_x^*\omega \in E^*(D(\xi)_x, S(\xi)_x)$ is a generator.

The key takeaway of [ABS64] is that, stably, $\text{Spin}_{\mathbb{F}}$ -bundles can be $K\mathbb{F}$ -oriented. For $\mathbb{F} = \mathbb{R}, \mathbb{C}$, this is true in the sense of an orientation as defined above, but for $\mathbb{F} = \mathbb{H}$, this is more subtle. What we get for $\mathbb{F} = \mathbb{H}$ is not an orientation, but rather a class in $K\mathbb{H}^*T\xi$. And in this case, multiplying by classes in $K\mathbb{R}^*X$ does not even give us an isomorphism of $K\mathbb{R}^*$ -modules in general (although in some sense it is close, see [Hu22]).

To define such an orientation, we need to consider more Clifford-algebraic structures varying over a space beyond Spin bundles. We would like to form bundles of Clifford algebras and modules.

4.2.1 K-Theory Classes from Clifford Bundles

Definition 4.2.3. Suppose ξ is a vector bundle equipped with a metric on a space X . Then the **Clifford algebra bundle** of ξ is the bundle $\text{Cl}(\xi)$ over X whose fiber over $x \in X$ is $\text{Cl}(\xi_x)$.

Note that we use the Clifford algebra of the negative of the quadratic form induced by the metric on ξ , where we assume the metric on ξ is positive-definite.

To specify the topology of this bundle, we use the typical method of patching together the bundle using trivializations over small neighborhoods and continuous transition maps. It is clear that the algebra structure is compatible with the topology, giving us continuous unit $1 : X \rightarrow \text{Cl}(\xi)$ and multiplication $m : \text{Cl}(\xi) \otimes \text{Cl}(\xi) \rightarrow \text{Cl}(\xi)$ maps. Of course, we can go on to construct the algebra bundles $\text{Cl}_{\mathbb{F}}(\xi) = \text{Cl}(\xi) \otimes \mathbb{F}$, which also admit continuous unit and multiplication maps. Moreover, the automorphism α and the antiautomorphisms $(-)^t$ and $\overline{(-)}$ define bundle maps $\text{Cl}_{\mathbb{F}}(\xi) \rightarrow \text{Cl}_{\mathbb{F}}(\xi)$. Using α , we see that the grading of a Clifford algebra extends to a grading of $\text{Cl}_{\mathbb{F}}(\xi)$, given by $\text{Cl}_{\mathbb{F}}(\xi) = \text{Cl}_{\mathbb{F}}(\xi)^0 \oplus \text{Cl}_{\mathbb{F}}(\xi)^1$.

Similarly, we can talk about module bundles over the Clifford algebra bundle.

Definition 4.2.4. If ξ is a vector bundle with metric, then a **graded Clifford module bundle** μ over $\text{Cl}_{\mathbb{F}}(\xi)$ is a vector bundle $\mu = \mu^0 \oplus \mu^1$ along with a map of vector bundles $\text{Cl}_{\mathbb{F}}(\xi) \otimes M \rightarrow M$ that defines a graded module over the algebra in each fiber.

Similar to our discussion of the K-theory of Clifford modules, we can define analogous groups $A_{\mathbb{F}}(n)$, defined in terms of graded Clifford module bundles over $\text{Cl}_{\mathbb{F}}(\xi)$. So, proceeding as we did before, let $M_{\mathbb{F}}(\xi)$ be the Grothendieck group of graded Clifford module bundles over $\text{Cl}_{\mathbb{F}}(\xi)$. Now, let $\xi \oplus 1$ be given a metric such that the trivial summand is orthogonal to ξ . Then we have a map of vector bundles $\text{Cl}_{\mathbb{F}}(\xi) \rightarrow \text{Cl}_{\mathbb{F}}(\xi \oplus 1)$ that is a map of Clifford algebras in each fiber, and therefore induces a restriction map $M_{\mathbb{F}}(\xi \oplus 1) \rightarrow M_{\mathbb{F}}(\xi)$. We define $A_{\mathbb{F}}(\xi)$ to be the cokernel of this map.

For the next step of the orientation, suppose that ξ is a vector bundle with metric on X . Suppose that μ is a graded module over $\text{Cl}_{\mathbb{F}}(\xi)$. Let $\pi : D(\xi) \rightarrow X$ be the projection of the unit disk bundle of ξ . Then μ pulls back to a bundle $\pi^*\mu$ on $D(\xi)$, and it is given by the

direct sum of the pullbacks of the graded parts of μ . Now, we have maps of vector bundles $\sigma_{\pm} : \pi^*\mu \rightarrow \pi^*\mu$ given by multiplying m in the fiber of $\pi^*\mu$ over $v \in D(\xi)$ by $\pm v \in \text{Cl}_{\mathbb{F}}(\xi)$. Note that since $\pm v$ is in the odd-graded part of $\text{Cl}_{\mathbb{F}}(\xi)$, σ_{\pm} maps $\pi^*\mu^0$ to $\pi^*\mu^1$ and vice versa.

Note that on $S(\xi)$, the maps σ_{\pm} are inverses. This is because $\sigma_{\pm}\sigma_{\mp}m = -v^2m = |v|^2m = m$. So, we have an exact sequence

$$0 \longrightarrow \pi^*\mu^1 \xrightarrow{\sigma_-} \pi^*\mu^0 \longrightarrow 0$$

on $S(\xi)$. Hence $\xi_{\mu} = (\pi^*\mu^1, \pi^*\mu_0, \sigma_-) \in \mathcal{C}^1(D(\xi), S(\xi))$ and we can form the difference bundle $\chi(\xi_{\mu}) \in K\mathbb{F}^0(D(\xi), S(\xi)) \cong \tilde{K}\mathbb{F}^0(T\xi)$.

To understand the multiplicative properties of this construction, we can use the formula for the product of difference bundles. However, this requires us to place a metric on the bundle $\pi^*\mu$ and then analyze the adjoint of σ_- . There is a particularly convenient choice:

Lemma 4.2.5. *If μ is a graded module over $\text{Cl}_{\mathbb{F}}(\xi)$, then there is a metric on μ that is invariant under the action of $\text{Pin}_{\mathbb{F}}(\xi)$. Under this metric, σ_- and σ_+ are adjoint.*

Proof. First, consider the restriction of all these bundles to a neighborhood on which ξ and μ are trivial. Then using the fact that for any (smooth) representation of a compact Lie group, a metric can be chosen that is invariant under the action of the group by integrating with respect to the Haar measure, we can find a metric on the restriction of μ to this neighborhood where the action of $\text{Pin}_{\mathbb{F}}(\xi)$ leaves the metric invariant. Then, using a partition of unity subordinate to this covering by neighborhoods where ξ and μ are trivial, we can sum up these metrics to get a metric on all of μ invariant under the action of $\text{Pin}_{\mathbb{F}}(\xi)$.

For the adjoint of σ_- , notice that $\frac{v}{|v|}$ is in $\text{Pin}_{\mathbb{F}}(\xi)$ for $v \in \xi$, provided $v \neq 0$. If $v = 0$, then $\sigma_{\pm} = 0$ on this fiber and therefore they are adjoint. If $v \neq 0$, then

$$\begin{aligned} (\sigma_-m, m') &= (-vm, m') \\ &= -|v| \left(\frac{v}{|v|}m, -\frac{v}{|v|^2}vm' \right) \\ &= -|v| \left(m, -\frac{v}{|v|}m' \right) \\ &= (m, vm') \\ &= (m, \sigma_+m'). \end{aligned} \quad \square$$

By the additivity and naturality of χ and the construction above, it follows that this defines a homomorphism $M_{\mathbb{F}}(\xi) \rightarrow \tilde{K}\mathbb{F}^0(T\xi)$. This induces a map from $A_{\mathbb{F}}(\xi)$:

Proposition 4.2.6. *Composing the maps $M_{\mathbb{F}}(\xi \oplus 1) \rightarrow M_{\mathbb{F}}(\xi)$ and $M_{\mathbb{F}}(\xi) \rightarrow \tilde{K}\mathbb{F}^0(T\xi)$ results in the zero map $M_{\mathbb{F}}(\xi \oplus 1) \rightarrow \tilde{K}\mathbb{F}^0(T\xi)$, and therefore there is an induced map $A_{\mathbb{F}}(\xi) \rightarrow \tilde{K}\mathbb{F}^0(T\xi)$.*

Proof. We want to show that σ_- extends to all of $D(\xi)$. By the construction above, σ_- extends to $S(\xi \oplus 1)$. Let $S_+(\xi \oplus 1) \subset S(\xi \oplus 1)$ be the subspace consisting of those vectors with nonnegative component in the trivial summand of $\xi \oplus 1$. Now, projecting $\xi \oplus 1$ onto ξ maps $S_+(\xi \oplus 1)$ homeomorphically onto $D(\xi)$, while preserving $S(\xi)$. Then $\sigma_-|_{S(\xi)}$ can be extended to an isomorphism $\pi^*\mu^1 \rightarrow \pi^*\mu^0$ on $D(\xi)$ by mapping $D(\xi)$ homeomorphically to $S_+(\xi \oplus 1)$ (since σ_- is defined on the pullbacks of μ to $D(\xi)$ and $D(\xi \oplus 1)$ along the projections down to X , we map the bundle on $D(\xi)$ to the bundle on $S_+(\xi \oplus 1)$ using the identity for individual vector spaces in the same fiber, which preserves the bundle on $S(\xi)$). Hence $\chi(\xi_\mu) = 0$ in this case. \square

These classes have good multiplicative properties. First, recall that the Thom space of the *external* Whitney sum of two vector bundles is homeomorphic to the smash product of the Thom spaces of the summands: $T(\xi_1 \oplus \xi_2) \cong T\xi_1 \wedge T\xi_2$. In this equation and what follows, we are making the pullbacks along the projections implicit so as to not clutter notation. For $(\mathbb{F}_1, \mathbb{F}_2, \mathbb{F})$ compatible for multiplication, this gives us a multiplication map $K\tilde{\mathbb{F}}_1(T\xi_1) \otimes K\tilde{\mathbb{F}}_2(T\xi_2) \rightarrow K\tilde{\mathbb{F}}(T(\xi_1 \oplus \xi_2))$.

There is also a map $M_{\mathbb{F}_1}(\xi_1) \otimes M_{\mathbb{F}_2}(\xi_2) \rightarrow M_{\mathbb{F}}(\xi_1 \oplus \xi_2)$, given by taking a module μ_1 over $\text{Cl}_{\mathbb{F}_1}(\xi_1)$ and a module μ_2 over $\text{Cl}_{\mathbb{F}_2}(\xi_2)$ and forming their external (graded) tensor product, which is a module over $\text{Cl}_{\mathbb{F}_1}(\xi_1) \otimes_{\mathbb{Z}/2\mathbb{Z}} \text{Cl}_{\mathbb{F}_2}(\xi_2) \cong \text{Cl}_{\mathbb{F}}(\xi_1 \oplus \xi_2)$. For the same reasons as we saw for the Clifford modules over Clifford algebras, this induces a map $A_{\mathbb{F}_1}(\xi_1) \otimes A_{\mathbb{F}_2}(\xi_2) \rightarrow A_{\mathbb{F}}(\xi_1 \oplus \xi_2)$.

Proposition 4.2.7. *The diagram*

$$\begin{array}{ccc} A_{\mathbb{F}_1}(\xi_1) \otimes A_{\mathbb{F}_2}(\xi_2) & \longrightarrow & A_{\mathbb{F}}(\xi_1 \oplus \xi_2) \\ \downarrow & & \downarrow \\ K\tilde{\mathbb{F}}_1(T\xi_1) \otimes K\tilde{\mathbb{F}}_2(T\xi_2) & \longrightarrow & K\tilde{\mathbb{F}}(T(\xi_1 \oplus \xi_2)) \end{array}$$

commutes.

Proof. Let μ_1 and μ_2 be graded $\text{Cl}_{\mathbb{F}_1}(\xi_1)$ - and $\text{Cl}_{\mathbb{F}_2}(\xi_2)$ -modules, respectively. We assume that they are given metrics invariant under their respective Pin bundles. If we first go to K-theory and then multiply, we get $\chi(\alpha)$, where $\alpha = (\alpha_1, \alpha_0, \sigma)$ and

$$\begin{aligned} \alpha_1 &= (\pi^*\mu_1^0 \otimes \pi^*\mu_2^1) \oplus (\pi^*\mu_1^1 \otimes \pi^*\mu_2^0), \\ \alpha_0 &= (\pi^*\mu_1^0 \otimes \pi^*\mu_2^0) \oplus (\pi^*\mu_1^1 \otimes \pi^*\mu_2^1), \\ \sigma &= \begin{bmatrix} \text{id} \otimes \sigma_- & \sigma_- \otimes \text{id} \\ \sigma_+ \otimes \text{id} & -\text{id} \otimes \sigma_+ \end{bmatrix}. \end{aligned}$$

These are bundles defined on $X = D(\xi_1) \times D(\xi_2)$ with σ being defined on $Y = D(\xi_1) \times S(\xi_2) \cup S(\xi_1) \times D(\xi_2)$. Letting $\alpha^i = \alpha_i$ (i.e. defining a graded bundle α with graded pieces α_0 and α_1), we see that α is just $\pi^*(\mu_1 \otimes_{\mathbb{Z}/2\mathbb{Z}} \mu_2)$. At a point of $(v, w) \in Y$, σ is given by

$$\sigma = \begin{bmatrix} -1 \otimes w & -v \otimes 1 \\ v \otimes 1 & -1 \otimes w \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \otimes w & v \otimes 1 \\ v \otimes 1 & -1 \otimes w \end{bmatrix}.$$

Now the matrix $\begin{bmatrix} 1 \otimes w & v \otimes 1 \\ v \otimes 1 & -1 \otimes w \end{bmatrix}$ gives the action of $v \otimes w \in \text{Cl}_{\mathbb{F}_1}(\xi_1) \otimes_{\mathbb{Z}/2\mathbb{Z}} \text{Cl}_{\mathbb{F}_2}(\xi_2)$ on $\pi^*(\mu_1 \otimes_{\mathbb{Z}/2\mathbb{Z}} \mu_2)$ (note the sign rule, giving the negative sign in the bottom right corner). So $\sigma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \sigma_-$.

Let $D_2(\xi_1 \oplus \xi_2)$ denote the disk bundle of radius two, and let $L(\xi_1 \oplus \xi_2)$ be the bundle of closed shells of inner radius one and outer radius two. Then the inclusions

$$\begin{aligned} i &: (D(\xi_1 \oplus \xi_2), S(\xi_1 \oplus \xi_2)) \rightarrow (D_2(\xi_1 \oplus \xi_2), L(\xi_1 \oplus \xi_2)), \\ j &: (D(\xi_1) \times D(\xi_2), D(\xi_1) \times S(\xi_2) \cup S(\xi_1) \times D(\xi_2)) \rightarrow (D_2(\xi_1 \oplus \xi_2), L(\xi_1 \oplus \xi_2)) \end{aligned}$$

are homotopy equivalences of pairs. Now, $\chi((\xi_1 \oplus \xi_2)_{\mu_1 \otimes_{\mathbb{Z}/2\mathbb{Z}} \mu_2})$ is pulled back along i to the difference bundle construction applied to $(\pi^*(\mu_1 \oplus \mu_2)^1, \pi^*(\mu_1 \oplus \mu_2)^0, \sigma_-)$ on $(D_2(\xi_1 \oplus \xi_2), L(\xi_1 \oplus \xi_2))$. However,

$$\chi(\pi^*(\mu_1 \oplus \mu_2)^1, \pi^*(\mu_1 \oplus \mu_2)^0, \sigma_-) = \chi\left(\pi^*(\mu_1 \oplus \mu_2)^1, \pi^*(\mu_1 \oplus \mu_2)^0, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \sigma_-\right)$$

on $(D_2(\xi_1 \oplus \xi_2), L(\xi_1 \oplus \xi_2))$ because there is an isomorphism of vector bundles appearing in the difference bundle construction given by multiplication by -1 on $\pi^*(\mu_1 \oplus \mu_2)^0$. Then, pulling back along j gives us $\chi(\alpha)$. And so the diagram commutes. \square

4.2.2 Clifford Bundles from Spin Bundles

The orientation class is a K-theory class on the Thom space, and we just saw how to form such classes using Clifford bundles. So now we need to form Clifford bundles from Spin bundles. One way to do this is to take a graded module over $\text{Cl}_{\mathbb{F}}(n)$ and “twist” it by a principal $\text{Spin}_{\mathbb{F}}(n)$ -bundle on X . If P is a principal $\text{Spin}_{\mathbb{F}}(n)$ -bundle on X , then let $\xi = P \times_{\text{Spin}_{\mathbb{F}}(n)} \mathbb{R}^n$ be the associated vector bundle (with metric inherited from \mathbb{R}^n). If M is a $\text{Cl}_{\mathbb{F}}(n)$ -module, then we can similarly form a vector bundle $\mu = P \times_{\text{Spin}_{\mathbb{F}}(n)} M$, where $\text{Spin}_{\mathbb{F}}(n)$ acts on the left on M by the inclusion $\text{Spin}_{\mathbb{F}}(n) \rightarrow \text{Cl}_{\mathbb{F}}(n)$.

Lemma 4.2.8. *Let P be a principal $\text{Spin}_{\mathbb{F}}(n)$ -bundle on X and let M be a graded $\text{Cl}_{\mathbb{F}}(n)$ -module. Then if we let $\xi = P \times_{\text{Spin}_{\mathbb{F}}(n)} \mathbb{R}^n$ and $\mu = P \times_{\text{Spin}_{\mathbb{F}}(n)} M$, then μ can be given the*

structure of a graded Clifford module bundle over $\text{Cl}_{\mathbb{F}}(\xi)$, where the multiplication is induced by

$$[x, v] \cdot [x, m] = [x, vm].$$

Proof. Suppose $U \subseteq X$ is an open set and we have a trivialization $\phi : P \rightarrow U \times \text{Spin}_{\mathbb{F}}(n)$. Then on U , this multiplication is induced by

$$((x, g), v) \cdot ((x, h), m) = ((x, g), vg^{-1}hm).$$

It is easy to see that this respects the equivalence relation for the balanced product $P \times_{\text{Spin}_{\mathbb{F}}(n)} M$, but for the first factor, it is important to understand that the way $\text{Spin}_{\mathbb{F}}(n)$ acts on \mathbb{R}^n and M are distinct and we are intertwining these different actions. For clarity, we denote the action of $\text{Spin}_{\mathbb{F}}(n)$ on \mathbb{R}^n by \star and recall that $a \star v = av a^{-1} \in \text{Cl}_{\mathbb{F}}(v)$. Then

$$\begin{aligned} ((x, ga), v) \cdot ((x, h), m) &= ((x, ga), va^{-1}g^{-1}hm) \\ &= ((x, g), av a^{-1}g^{-1}hm), \\ ((x, g), a \star v) \cdot ((x, h), m) &= ((x, g), (a \star v)g^{-1}hm) \\ &= ((x, g), av a^{-1}g^{-1}hm). \end{aligned}$$

So the multiplication respects the equivalence relation of the balanced product $P \times_{\text{Spin}_{\mathbb{F}}(n)} \mathbb{R}^n$ as well. It is easy to see that these multiplications agree on the intersections for various trivializing neighborhoods U and thus define a map of vector bundles $\xi \otimes \mu \rightarrow \mu$. This then clearly extends to a map $\text{Cl}_{\mathbb{F}}(\xi) \otimes \mu \rightarrow \mu$ which makes μ into a graded Clifford module bundle. \square

Combining the two constructions above, we see that if ξ is a principal $\text{Spin}_{\mathbb{F}}(n)$ -bundle then we can take any $\text{Cl}(n)_{\mathbb{F}}$ -module and map it to a class in $\tilde{K}\mathbb{F}^0(T\xi)$. This is where the orientation will come from; if we can choose a $\text{Cl}(n)_{\mathbb{F}}$ -module such that this construction applied to a bundle over a point always gives a generator of $\tilde{K}\mathbb{F}^0(S^n)$ (or something appropriate when $\mathbb{F} = \mathbb{H}$), then we can define the orientation class to be its image in $\tilde{K}\mathbb{F}^0(T\xi)$.

Clearly this construction defines a homomorphism of abelian groups $M_{\mathbb{F}}(n) \rightarrow \tilde{K}\mathbb{F}^0(T\xi)$ factoring through $M_{\mathbb{F}}(\xi)$. This factors through $A_{\mathbb{F}}(n)$, since if M is a graded $\text{Cl}_{\mathbb{F}}(n+1)$ -bundle, then we can define a module μ over $\text{Cl}_{\mathbb{F}}(\xi \oplus 1)$ that restricts to the right module over $\text{Cl}_{\mathbb{F}}(\xi)$ by using the action of $\text{Spin}_{\mathbb{F}}(n)$ on $\mathbb{R}^{n+1} \cong \mathbb{R}^n \oplus \mathbb{R}$ that is given by the standard representation on \mathbb{R}^n and the trivial representation on \mathbb{R} , and then using the multiplication $[x, v] \cdot [x, m] = [x, vm]$. Then $P \times_{\text{Spin}_{\mathbb{F}}(n)} \mathbb{R}^{n+1} \cong \xi \oplus 1$ and we get the desired $\text{Cl}_{\mathbb{F}}(\xi \oplus 1)$ -module bundle $\mu = P \times_{\text{Spin}_{\mathbb{F}}(n)} M$. Let $\beta_P : A_{\mathbb{F}}(n) \rightarrow A_{\mathbb{F}}(\xi)$ denote the map taking the class of M to the class of $P \times_{\text{Spin}_{\mathbb{F}}(n)} M$, and let $\alpha_P : A_{\mathbb{F}}(n) \rightarrow \tilde{K}\mathbb{F}^0(T\xi)$ denote the composite map.

However, the maps $A_{\mathbb{F}}(n) \rightarrow \tilde{K}\mathbb{F}^0(T\xi)$ admit even more structure in the form of multiplication. If we have a $\text{Spin}_{\mathbb{F}_1}(n_1)$ -bundle P_1 and a $\text{Spin}_{\mathbb{F}_2}(n_2)$ -bundle P_2 , we can obtain a $\text{Spin}_{\mathbb{F}}(n_1 + n_2)$ -bundle P by forming the balanced product

$$P := (P_1 \times P_2) \times_{\text{Spin}_{\mathbb{F}_1}(n_1) \times \text{Spin}_{\mathbb{F}_2}(n_2)} \text{Spin}_{\mathbb{F}}(n_1 + n_2).$$

Here the left action of $\text{Spin}_{\mathbb{F}_1}(n_1) \times \text{Spin}_{\mathbb{F}_2}(n_2)$ on $\text{Spin}_{\mathbb{F}}(n_1 + n_2)$ is given by the homomorphism $\text{Spin}_{\mathbb{F}_1}(n_1) \times \text{Spin}_{\mathbb{F}_2}(n_2) \rightarrow \text{Spin}_{\mathbb{F}}(n_1 + n_2)$.

Proposition 4.2.9. *If M_1 and M_2 are $\text{Cl}_{\mathbb{F}_1}(n_1)$ - and $\text{Cl}_{\mathbb{F}_2}(n_2)$ -modules, respectively, then*

$$\beta_P(M_1 \otimes_{\mathbb{Z}/2\mathbb{Z}} M_2) = \beta_P(M_1)\beta_P(M_2).$$

Proof. We show that the representing Clifford module bundles are isomorphic. An isomorphism from the bundle representing $\beta_P(M_1)\beta_P(M_2)$ to the bundle representing $\beta_P(M_1 \otimes_{\mathbb{Z}/2\mathbb{Z}} M_2)$ is

$$[p_1, m_1] \otimes [p_2, m_2] \mapsto [[(p_1, p_2), 1], m_1 \otimes m_2].$$

It is easy to see that this is well-defined and defines an isomorphism of vector bundles. And it is a homomorphism of module bundles, since

$$\begin{aligned} & ([p_1, v_1] \otimes [p_2, v_2])([p_1, m_1] \otimes [p_2, m_2]) \\ &= (-1)^{|m_1|} [p_1, v_1 m_1] \otimes [p_2, v_2 m_2] \\ &\mapsto [[(p_1, p_2), 1], (-1)^{|m_1|} v_1 m_1 \otimes v_2 m_2] \\ &= [[(p_1, p_2), 1], (v_1 \otimes v_2)(m_1 \otimes m_2)] \\ &= [[(p_1, p_2), 1], v_1 \otimes v_2] \cdot [[(p_1, p_2), 1], m_1 \otimes m_2]. \quad \square \end{aligned}$$

It then follows that $\alpha_P(M_1 \otimes_{\mathbb{Z}/2\mathbb{Z}} M_2) = \alpha_P(M_1)\alpha_P(M_2)$. If we consider the unique $\text{Spin}_{\mathbb{F}}(n)$ -bundles over a point, this gives a multiplicative homomorphism from $A_{\mathbb{F}}^*$ to $\tilde{K}\mathbb{F}^0(S^*) \cong \pi_* K\mathbb{F}$.

Proposition 4.2.10. *The maps $\alpha_{\mathbb{F}} : A_{\mathbb{F}}^* \rightarrow \pi_* K\mathbb{F}$ are ring homomorphisms in the case $\mathbb{F} = \mathbb{C}, \mathbb{H}$ and a homomorphism of $A_{\mathbb{R}}^*$ -modules when $\mathbb{F} = \mathbb{H}$.*

Proof. For $n = 0$ and $\mathbb{F} = \mathbb{R}, \mathbb{C}$ in particular, the unit $[\mathbb{F}] = 1 \in A_{\mathbb{F}}^0$ maps to $1 \in \tilde{K}\mathbb{F}_*^0$, since the corresponding module (bundle) is μ , where $\mu^1 = 0$ and $\mu^0 = \mathbb{F}$, and in $K\mathbb{F}^0(* \coprod *)$, we have $\check{\xi}_{\mu} - \pi_1^* \mu_1$ given by the bundle that is \mathbb{F} on the point indexed by 0 and 0 on the point indexed by 1 (which is the ∞ of the Thom space). So the only element of $K\mathbb{F}^0(* \coprod *, *_1)$ that would map to this is the unit $1 \in K\mathbb{F}^0(* \coprod *, *_1)$, and therefore the difference bundle must be the unit of $K\mathbb{F}_*^0$. Therefore the maps $\alpha_{\mathbb{F}} : A_{\mathbb{F}}^* \rightarrow \pi_* K\mathbb{F}$ are ring homomorphisms for $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and a module homomorphism for $\mathbb{F} = \mathbb{H}$. \square

Now let us discuss the correct choice of the module that generates the K-theory of the sphere.

Theorem 4.2.11. *The maps $\alpha_{\mathbb{F}} : A_{\mathbb{F}}^* \rightarrow \pi_{\geq 0} K\mathbb{F}$ are isomorphisms.*

Proof. For the cases $\mathbb{F} = \mathbb{R}, \mathbb{C}$, we know that these two rings are abstractly isomorphic, so it will suffice to show that $\alpha_{\mathbb{F}}$ sends generators to corresponding generators. Let's begin with $\mathbb{F} = \mathbb{C}$. If we consider the difference bundle construction, we see that $\alpha_{\mathbb{C}}(\mu_2)$ corresponds to the reduction of a bundle on S^2 given by two trivial bundles (of \mathbb{C} -dimension 1) on D^2 glued together along S^1 using the action of $-v$ on μ_2 , where $v \in S^1$ is considered as an element of $\text{Cl}_{\mathbb{C}}(2)$. So μ_2 , the generator of $A_{\mathbb{C}}^2$, maps to x_2 , the reduced canonical bundle over $\mathbb{C}\mathbb{P}^1$.

Now we consider the real case. Under complexification, v_8 goes to x_2^4 , so we see that $\alpha_{\mathbb{R}}(\lambda_8)$ must go to the class of x_2^4 under complexification, as the complexification of λ_8 is $\mu_2^4 \in K_{\mathbb{C}}^{-8}$. Therefore it must be the case that $\alpha_{\mathbb{R}}(\lambda_8) = v_8$. For $\mu_4 \in A_{\mathbb{R}}^4$, we see that $\alpha_{\mathbb{R}}(\mu_4)^2 = 4v_8$. And so $\alpha_{\mathbb{R}}(\mu_4) = \pm x_4$, since no other elements go to $4v_8$ after being squared. Finally, for $\xi_1 \in A_{\mathbb{R}}^1$, we use the fact that $\eta_1 \in K\mathbb{R}^{-1}(*)$ is represented by the canonical bundle on $\mathbb{R}\mathbb{P}^1 \cong S^1$. Since $\text{Cl}(1) \cong \mathbb{C}$, and therefore ξ_1 is represented by the irreducible module \mathbb{C} . The unit sphere in \mathbb{R} is $\pm e_1$, and multiplication by these corresponds to multiplication by $\pm i$. So if we glue two copies of the unit disk $[-1, 1]$ together along the unit sphere, with a one-dimensional trivial vector bundle on one copy given by $\mathbb{R} \subset \mathbb{C}$ and $\mathbb{R}i \subset \mathbb{C}$ on the other, with the one on the unit sphere given by multiplication by $\pm i$, we see that the resulting bundle is the Möbius bundle, which is the canonical bundle on $\mathbb{C}\mathbb{P}^1$.

For the quaternionic case, we need to show that the generators of $A_{\mathbb{H}}^*$ in degrees zero and four go to generators of $K\tilde{\mathbb{H}}^{\leq 0}(*)$ in degrees zero and -4 . For degree 0, the proof of this is analogous to the proof that $\alpha_{\mathbb{F}}(1) = 1$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$. For degrees ± 4 , this follows from the fact that the diagram

$$\begin{array}{ccc} A_{\mathbb{H}}^4 & \xrightarrow{\alpha_{\mathbb{H}}} & K\tilde{\mathbb{F}}_*^{-4} \\ \downarrow & & \downarrow \\ A_{\mathbb{R}}^4 & \xrightarrow{\alpha_{\mathbb{R}}} & K\tilde{\mathbb{R}}_*^{-4} \end{array}$$

commutes, where the vertical maps are restriction along $\mathbb{R} \rightarrow \mathbb{H}$. The bottom arrow is an isomorphism as we saw above, and the rightmost map is an isomorphism [Bot59, p. 3.14]. The left arrow is also an isomorphism, and this can be seen by comparing the dimension over \mathbb{R} of modules representing generators of $A_{\mathbb{H}}^4$ and $A_{\mathbb{R}}^4$. So $\alpha_{\mathbb{H}}$ must be an isomorphism for these degrees as well. This argument is due to [Hu22]. \square

Theorem 4.2.12. *Any $\text{Spin}_{\mathbb{R}}(8n)$ -bundle is $K\mathbb{R}$ -orientable, and any $\text{Spin}_{\mathbb{C}}(2n)$ -bundle is $K\mathbb{C}$ -orientable.*

Proof. For \mathbb{R} , the orientation is given by $\alpha_{\mathbb{R}}(v_8^n)$, and for \mathbb{C} , it is given by $\alpha_{\mathbb{C}}(\mu_2)$. \square

For \mathbb{H} , we see that for a $\text{Spin}_{\mathbb{H}}(8n)$ -bundle ξ , there is an “orientation class” $\alpha_{\mathbb{H}}(\mu_0)$ in $K\tilde{\mathbb{H}}^0(T\xi)$. This class is notably *not* a generator of $K\tilde{\mathbb{H}}^0(S^{8n})$ for each fiber (unless we invert 2).

4.3 Stabilization

The orientations and the maps used to define them enjoy many properties such as naturality and some multiplicative relations. A natural expectation is that there should be a “universal orientation” that captures all of these phenomena, and any orientation should factor through it. Clearly this should be related to the classifying spaces $B\text{Spin}_{\mathbb{F}}(n)$, which support the universal $\text{Spin}_{\mathbb{F}}(n)$ -bundles that all other bundles are pulled back from. It should also involve the Thom spaces of the universal bundles, since that is where the orientations live.

Using the maps $\text{Spin}_{\mathbb{F}}(n) \rightarrow \text{Spin}_{\mathbb{F}}(n+1)$ which fit into diagrams

$$\begin{array}{ccc} \text{Spin}_{\mathbb{F}}(n) & \longrightarrow & \text{Spin}_{\mathbb{F}}(n+1) \\ \downarrow & & \downarrow \\ \text{SO}(n) & \longrightarrow & \text{SO}(n+1) \end{array}$$

that commute, we get commutative diagrams

$$\begin{array}{ccc} B\text{Spin}_{\mathbb{F}}(n) & \longrightarrow & B\text{Spin}_{\mathbb{F}}(n+1) \\ \downarrow & & \downarrow \\ B\text{SO}(n) & \longrightarrow & B\text{SO}(n+1) \end{array}$$

So any principal $\text{Spin}_{\mathbb{F}}(n)$ -bundle can be made into a principal $\text{Spin}_{\mathbb{F}}(n+1)$ -bundle, and taking the induced vector bundles adding a trivial summand. If $\gamma_{\mathbb{F},n}$ is the vector bundle on the classifying space $B\text{Spin}_{\mathbb{F}}(n)$, then this implies that $\gamma_{\mathbb{F},n} \oplus 1$ is pulled back from $\gamma_{\mathbb{F},n+1}$. Taking Thom spaces, this induces a map $T(\gamma_{\mathbb{F},n} \oplus 1) \cong \Sigma T\gamma_{\mathbb{F},n} \rightarrow \gamma_{\mathbb{F},n+1}$.

Definition 4.3.1. The **Thom spectrum** $M\text{Spin}_{\mathbb{F}}$ is defined to be the spectrum with n th space $T\gamma_{\mathbb{F},n}$ and structure maps $\Sigma T\gamma_{\mathbb{F},n} \rightarrow T\gamma_{\mathbb{F},n+1}$ as above.

We can loosely think of this as the “universal $\text{Spin}_{\mathbb{F}}$ Thom space”, but we are now in the world of stable homotopy theory. For any $\text{Spin}_{\mathbb{F}}(n)$ -bundle, we can form the Thom spectrum $M\xi$ of the associated vector bundle ξ with k th space given by $\Sigma^{k-n}T\xi$ (thought of as the

Thom space of $\xi \oplus (k - n)$). The classifying map of the bundle then induces a map of Thom spectra $M\xi \rightarrow M\text{Spin}_{\mathbb{F}}$.

These spectra have additional multiplicative structure derived from $\text{Spin}_{\mathbb{F}}$ -structures on direct sums. For compatible fields, the pairings of $\text{Spin}_{\mathbb{F}}$ groups in Proposition 3.3.9 give us commutative diagrams

$$\begin{array}{ccc} B\text{Spin}_{\mathbb{F}_1}(n_1) \times B\text{Spin}_{\mathbb{F}_2}(n_2) & \longrightarrow & B\text{Spin}_{\mathbb{F}}(n_1 + n_2) \\ \downarrow & & \downarrow \\ B\text{SO}(n_1) \times B\text{SO}(n_2) & \longrightarrow & B\text{SO}(n_1 + n_2) \end{array}$$

These maps are associative and commutative (for \mathbb{R}, \mathbb{C}) in a way that commutes with the associativity and commutativity data for $B\text{SO}(n)$. Taking Thom spaces, this gives us maps $T\gamma_{\mathbb{F}_1, n_1} \wedge T\gamma_{\mathbb{F}_2, n_2} \rightarrow T\gamma_{\mathbb{F}, n_1 + n_2}$. Putting all of these maps together at the level of spectra gives $M\text{Spin}_{\mathbb{R}}$ and $M\text{Spin}_{\mathbb{C}}$ the structure of commutative ring spectra, while $M\text{Spin}_{\mathbb{H}}$ has the structure of a module spectrum over $M\text{Spin}_{\mathbb{R}}$.

Now the universal Atiyah–Bott–Shapiro orientations should be maps $M\text{Spin}_{\mathbb{F}} \rightarrow K\mathbb{F}$. To define such a map, consider the orientation for the universal bundle on $B\text{Spin}_{\mathbb{F}}(n)$, which is classified by a map $T\gamma_{\mathbb{F}, n} \rightarrow B\text{GL}_{\mathbb{F}}$. Using the loops-suspension adjunction, we can use this to define the orientation map for levels less than or equal to n in $M\text{Spin}_{\mathbb{F}}$ and $K\mathbb{F}$. So we can define arbitrarily large finite pieces of the orientation map. To get the entire map, we need to show that these finite pieces agree. We need to show that the orientations for $\gamma_{\mathbb{F}, n}$ and $\gamma_{\mathbb{F}, n} \oplus P_{\mathbb{F}}$ agree after applying Bott periodicity. However, this is true because the orientation class on $T(\gamma_{\mathbb{F}, n} \oplus P_{\mathbb{F}}) \cong \Sigma^{P_{\mathbb{F}}} T\gamma_{\mathbb{F}, n}$ is given by multiplying the orientation class on $T\gamma_{\mathbb{F}, n}$ by the orientation class for the unique $\text{Spin}_{\mathbb{F}}(P_{\mathbb{F}})$ -bundle over a point. But this class is the Bott class, which gives the Bott periodicity isomorphism by multiplication. So by choosing appropriate homotopies, we indeed get maps $M\text{Spin}_{\mathbb{F}} \rightarrow K\mathbb{F}$. Moreover, these maps are multiplicative in the sense appropriate for \mathbb{F} , due to Proposition 4.2.9.

Theorem 4.3.2. *There are maps $\varphi_{\mathbb{F}} : M\text{Spin}_{\mathbb{F}} \rightarrow K\mathbb{F}$ with the following properties:*

- For $\mathbb{F} = \mathbb{R}, \mathbb{C}$, $\varphi_{\mathbb{F}}$ is a map of ring spectra.
- The map $\varphi_{\mathbb{H}}$ is a map of $M\text{Spin}$ -module spectra.
- If ξ is a $\text{Spin}(8n)$ -bundle, then the composite map $M\xi \rightarrow M\text{Spin} \rightarrow K\mathbb{R}$ classifies the orientation class defined earlier.
- If ξ is a $\text{Spin}_{\mathbb{C}}(2n)$ -bundle, then the composite $M\xi \rightarrow M\text{Spin}_{\mathbb{C}} \rightarrow K\mathbb{C}$ classifies the orientation class of $T\xi$.

- If ξ is a $\mathrm{Spin}_{\mathbb{H}}(8n)$ -bundle, then the composition $M\xi \rightarrow M\mathrm{Spin}_{\mathbb{H}} \rightarrow K\mathbb{H}$ classifies the “orientation class” we defined in $\tilde{K}\mathbb{H}^0(T\xi)$.

These maps of spectra have even nicer properties that we will not discuss in detail. One of these features is that the Thom spectra $M\mathrm{Spin}_{\mathbb{F}}$ and the K-theory spectra $K\mathbb{F}$ both admit higher homotopical coherences for their ring/module structures in the form of E_{∞} -structures. Then, one can show that the Atiyah–Bott–Shapiro orientation preserves this structure for $\mathbb{F} = \mathbb{R}, \mathbb{C}$ (see [Joa04]). For $\mathbb{F} = \mathbb{H}$, the author believes this to be true, but do not know of a proof yet.

Chapter 5

The Anderson–Brown–Peterson Splitting

Using the Atiyah–Bott–Shapiro orientations together with standard tools of homotopy theory, it is possible to completely determine the structure of the $\text{Spin}_{\mathbb{F}}$ bordism groups. To do this, we completely determine $M\text{Spin}_{\mathbb{F}}$ in terms of simpler spectra, whose homotopy groups are known, 2-locally. Then, using the fact that the homotopy groups of $M\text{Spin}_{\mathbb{F}}$ do not contain any odd torsion, we can easily determine the homotopy groups of $M\text{Spin}_{\mathbb{F}}$, and therefore the $\text{Spin}_{\mathbb{F}}$ bordism groups, up to isomorphism.

5.1 Spin Bordism and Thom Spectra

The structure of a Spin bundle is particularly applicable to manifolds. For technical reasons, instead of treating manifolds with a Spin structure on their tangent bundle, it is customary in bordism theory to treat manifolds with Spin structures on their stable normal bundle.

Theorem 5.1.1 (Hirsch). *If M is a manifold (with or without boundary), then for k sufficiently large, any two embeddings $M \rightarrow \mathbb{R}^k$ are regularly homotopic (homotopic through immersions, with the homotopy inducing a homotopy of maps between tangent bundles). Moreover, the homotopy is unique up to a homotopy of homotopies fixing the endpoints.*

Proof. See [Hir59]. □

So, choose an embedding $M \rightarrow \mathbb{R}^k$. If $\nu_k : M \rightarrow BO(n - k)$ is the map classifying the normal bundle of this embedding, then we get a map $\nu : M \rightarrow BO$ by composing with $BO(n - k) \rightarrow BO$. By the theorem above, for any other embedding $M \rightarrow \mathbb{R}^{k'}$, if we form the map $\nu' : M \rightarrow BO$ using the same procedure, then we can find a homotopy from ν to ν' . Moreover, these homotopies compose up to a homotopy of homotopies fixing the endpoints. The map ν will be referred to as the **stable normal bundle**.

From now on, we will assume that the map $B\text{Spin}_{\mathbb{F}} \rightarrow BO$ is a Serre fibration. Since we can always replace any map by a fibration up to homotopy, there is no loss of generality in doing this.

Definition 5.1.2. Let M be a manifold. Then a **normal $\text{Spin}_{\mathbb{F}}$ -structure** on M is a homotopy class of maps $M \rightarrow B\text{Spin}_{\mathbb{F}}$ lifting ν .

Specifically, we require that the maps $M \rightarrow B\text{Spin}_{\mathbb{F}}$ lift ν in the category of topological spaces and continuous maps, and that the homotopies are homotopies through such lifts.

For now, a $\text{Spin}_{\mathbb{F}}$ -structure will refer to a normal $\text{Spin}_{\mathbb{F}}$ -structure, and a $\text{Spin}_{\mathbb{F}}$ -manifold is a manifold together with a normal $\text{Spin}_{\mathbb{F}}$ -structure.

Since the map $B\text{Spin}_{\mathbb{F}} \rightarrow BO$ is a fibration, if $\nu' : M \rightarrow BO$ is another map homotopic to ν and $f : M \rightarrow B\text{Spin}_{\mathbb{F}}$ represents a normal $\text{Spin}_{\mathbb{F}}$ -structure on M , then we can use the homotopy from ν to ν' and the homotopy lifting property of the map $B\text{Spin}_{\mathbb{F}} \rightarrow BO$ to find a map $f' : M \rightarrow B\text{Spin}_{\mathbb{F}}$ that lifts ν' .

If N is a $\text{Spin}_{\mathbb{F}}$ -manifold with boundary, then trivializing the normal bundle of $\partial N \subset N$ so that the constant section at 1 of the trivialized normal bundle $\partial N \times \mathbb{R}$ consists of inward-pointing normal vectors, then using the $\text{Spin}_{\mathbb{F}}$ -structure on N and the homotopy lifting procedure above, we get an induced $\text{Spin}_{\mathbb{F}}$ -structure on ∂N . This is done by embedding N in \mathbb{R}^{k+1} such that ∂N embeds in \mathbb{R}^k , with a collar neighborhood U of ∂N meeting \mathbb{R}^k orthogonally at ∂N . This embedding is such that the trivializing section of normal vectors of ∂N in N all have positive first coordinate in $\mathbb{R}^{k+1} \cong \mathbb{R} \times \mathbb{R}^k$.

In particular, if M is a manifold without boundary equipped with a $\text{Spin}_{\mathbb{F}}$ -structure, then we get a $\text{Spin}_{\mathbb{F}}$ -structure on $M \times I$ using the projection $M \times I \rightarrow M$ that is a homotopy equivalence. Then, by considering the inclusion $M \rightarrow M \times \{1\}$, we get another $\text{Spin}_{\mathbb{F}}$ -structure on M , denoted $-M$.

Definition 5.1.3. Let M_0 and M_1 be closed $\text{Spin}_{\mathbb{F}}$ -manifolds. Then a **bordism** from M_0 to M_1 is a compact manifold N and a $\text{Spin}_{\mathbb{F}}$ -structure preserving diffeomorphism $M_0 \amalg -M_1 \rightarrow N$.

We say that two manifolds are **bordant** if there exists a bordism between them.

Lemma 5.1.4. *Bordism is an equivalence relation on closed $\text{Spin}_{\mathbb{F}}$ -manifolds.*

Proof. See [Sto68, p. 5]. □

Proposition 5.1.5. *The disjoint union of $\text{Spin}_{\mathbb{F}}$ -manifolds gives $\Omega_*^{\text{Spin}_{\mathbb{F}}}$ the structure of a graded abelian group. For $\mathbb{F} = \mathbb{R}, \mathbb{C}$, the product of $\text{Spin}_{\mathbb{F}}$ -manifolds makes these groups into graded rings. And for $\mathbb{F} = \mathbb{H}$, the product of $\text{Spin}_{\mathbb{R}}$ - and $\text{Spin}_{\mathbb{H}}$ -manifolds makes $\Omega_*^{\text{Spin}_{\mathbb{H}}}$ into a module over $\Omega_*^{\text{Spin}_{\mathbb{R}}}$.*

Proof. The abelian group structure on bordism groups is automatic for any bordism theory. The products are induced by the maps $B\text{Spin}_{\mathbb{F}}(n) \times B\text{Spin}_{\mathbb{F}}(m) \rightarrow B\text{Spin}_{\mathbb{F}}(n+m)$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and $B\text{Spin}_{\mathbb{R}}(n) \times B\text{Spin}_{\mathbb{H}}(m) \rightarrow B\text{Spin}_{\mathbb{H}}(n+m)$ for \mathbb{H} . As we saw previously in our discussion of the products for the Thom spectra $M\text{Spin}_{\mathbb{F}}$, these maps commute with the products $BO(n) \times BO(m) \rightarrow BO(n+m)$. Again, referencing [Sto68, p. 24-25], we see that this induces ring/module structures. \square

The reason it is possible to calculate the structure of these bordism groups, and even some of the ring structure, is due to the **Pontryagin–Thom isomorphism**:

Theorem 5.1.6. *There are isomorphisms of graded abelian groups*

$$\Omega_*^{\text{Spin}_{\mathbb{F}}} \cong \pi_* M\text{Spin}_{\mathbb{F}}. \quad (5.1)$$

If $\mathbb{F} = \mathbb{R}, \mathbb{C}$, then this is an isomorphism of graded rings. For $\mathbb{F} = \mathbb{H}$, then this (along with the isomorphism for $\mathbb{F} = \mathbb{R}$) gives an isomorphism of $\Omega_^{\text{Spin}_{\mathbb{R}}}$ -modules.*

Proof. The general Pontryagin-Thom isomorphism (see [Sto68, p. 18]) gives the result. \square

5.2 Cohomology of Classifying Spaces and Thom Spectra

So, to understand the 2-local structure of $M\text{Spin}_{\mathbb{F}}$, it will suffice to understand their $\mathbb{Z}/2\mathbb{Z}$ -cohomology, as they have finitely generated homotopy groups. And to determine this structure, we will make heavy use of the mod 2 Steenrod algebra \mathcal{A} . Unless otherwise stated, all cohomology in this section will be taken with $\mathbb{Z}/2\mathbb{Z}$ coefficients. Also, we will be working in the stable homotopy class, so a “map” will mean a homotopy class of morphisms of spectra. To begin with, we discuss the cohomology of the relevant classifying spaces.

Proposition 5.2.1. *There are isomorphisms*

$$\begin{aligned} H^* B\text{Spin} &\cong \mathbb{Z}/2\mathbb{Z}[w_i \mid i \geq 2, i \neq 2^k + 1 \text{ for } k \geq 0], \\ H^* B\text{Spin}_{\mathbb{C}} &\cong \mathbb{Z}/2\mathbb{Z}[w_i \mid i \geq 2, i \neq 2^{k+1} + 1 \text{ for } k \geq 0], \\ H^* B\text{Spin}_{\mathbb{H}} &\cong \mathbb{Z}/2\mathbb{Z}[w_i \mid i \geq 2, i \neq 2^{k+2} + 1 \text{ for } k \geq 0]. \end{aligned}$$

Here w_i denotes the i th Stiefel-Whitney class of the stable vector bundle on $B\text{Spin}_{\mathbb{F}}$ classified by the map $B\text{Spin}_{\mathbb{F}} \rightarrow BSO$.

Proof. Recall that there are homotopy fiber sequences $BN_{\mathbb{F}} \rightarrow B\text{Spin}_{\mathbb{F}} \rightarrow BSO$. If we apply the Serre spectral sequence to this fibration, we get a cohomologically graded spectral sequence with E_2 page

$$E_2^{p,q} \cong H^p(BSO; H^q BN_{\mathbb{F}})$$

converging to $H^*B\text{Spin}_{\mathbb{F}}$. One can compute the transgression of this spectral sequence, and, using the fact that the transgression commutes with the action of the Steenrod algebra, completely determine the result that it converges to. \square

Using this result, we can also determine the cohomology of the Thom spectrum $M\text{Spin}_{\mathbb{F}}$ with the Thom isomorphism. We have:

$$\begin{aligned} H^*M\text{Spin} &\cong \mathbb{Z}/2\mathbb{Z}[w_i \mid i \geq 2, i \neq 2^k + 1 \text{ for } k \geq 0]U, \\ H^*M\text{Spin}_{\mathbb{C}} &\cong \mathbb{Z}/2\mathbb{Z}[w_i \mid i \geq 2, i \neq 2^{k+1} + 1 \text{ for } k \geq 0]U_{\mathbb{C}}, \\ H^*M\text{Spin}_{\mathbb{H}} &\cong \mathbb{Z}/2\mathbb{Z}[w_i \mid i \geq 2, i \neq 2^{k+2} + 1 \text{ for } k \geq 0]U_{\mathbb{H}}. \end{aligned}$$

The action of the Steenrod algebra is determined by the Thom isomorphism as well, using the Cartan formula and the rule $\text{Sq}^i U_{\mathbb{F}} = w_i U_{\mathbb{F}}$, where $U_{\mathbb{F}}$ is the Thom class. Now we have a complete description of the cohomology of $M\text{Spin}_{\mathbb{F}}$, including its Steenrod module structure, which is the first step to determining the 2-local structure. From here, it is possible to determine the homotopy groups of $M\text{Spin}_{\mathbb{F}}$ using the Adams spectral sequence. However, we can do much more by showing that $M\text{Spin}_{\mathbb{F}}$ decomposes into a wedge sum of simpler spectra once we localize at 2. To do this, we will need to understand the cohomology of these summands as well. Luckily these are quite a bit simpler in terms of the Steenrod algebra.

One construction of stable homotopy that we need to understand the spectra in the splitting is the *connective cover* construction. We briefly recall it here.

Proposition 5.2.2. *Let X be a spectrum. Then for any $n \in \mathbb{Z}$, there exists a spectrum $X\langle n \rangle$ called the n -**connective cover** of X together with a map $X\langle n \rangle \rightarrow X$, such that $\pi_k X\langle n \rangle \cong 0$ if $k < n$, and the map $X\langle n \rangle \rightarrow X$ induces isomorphisms on π_k for $k \geq n$.*

Additionally, $X\langle n \rangle$ has the universal property that if Y is a spectrum with $\pi_k Y$ trivial for $k < n$, then for any map $f : Y \rightarrow X$, there is a unique map $\tilde{f} : Y \rightarrow X\langle n \rangle$ such that the diagram

$$\begin{array}{ccc} & & X\langle n \rangle \\ & \nearrow \tilde{f} & \downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

commutes.

For some spectra, such as the K-theory spectra, it is customary to denote the 0-connective cover by making the symbol we use to denote the spectrum lowercase, i.e. $K\mathbb{F}\langle 0 \rangle = k\mathbb{F}$. Note that a similar construction exists for spaces and their homotopy groups as well.

Proposition 5.2.3. *We have the following isomorphisms of Steenrod modules:*

$$H^*HZ/2\mathbb{Z} \cong \mathcal{A},$$

$$\begin{aligned}
H^*k\mathbb{R}\langle 8k \rangle &\cong \Sigma^{8k}\mathcal{A}/(\mathcal{ASq}^1 + \mathcal{ASq}^2), \\
H^*k\mathbb{R}\langle 8k + 2 \rangle &\cong \Sigma^{8k+2}\mathcal{A}/\mathcal{ASq}^3, \\
H^*k\mathbb{C}\langle 2k \rangle &\cong \Sigma^{2k}\mathcal{A}/(\mathcal{ASq}^1 + \mathcal{ASq}^3), \\
H^*k\mathbb{H}\langle 8k \rangle &\cong \Sigma^{8k}\mathcal{A}/(\mathcal{ASq}^1 + \mathcal{ASq}^5), \\
H^*k\mathbb{H}\langle 8k + 4 \rangle &\cong \Sigma^{8k+4}\mathcal{A}/(\mathcal{ASq}^1 + \mathcal{ASq}^2).
\end{aligned}$$

Proof. By the Yoneda lemma and the fact that $H\mathbb{Z}/2\mathbb{Z}$ represents cohomology, we have $H^*H\mathbb{Z}/2\mathbb{Z} \cong \mathcal{A}$. For $H^*k\mathbb{R}\langle n \rangle$ and $H^*k\mathbb{C}\langle n \rangle$, the main results of [Sto63] describe the cohomology of the level spaces that make up these spectra in low degrees, and the cohomology of the spectra can be found by taking a suitable colimit. The cohomology of $k\mathbb{H}\langle n \rangle$ can be found from the cohomology of $k\mathbb{R}\langle n \rangle$ and using the isomorphisms $\Sigma^4k\mathbb{H}\langle n \rangle \cong k\mathbb{R}\langle n + 4 \rangle$, which are induced by the isomorphism $\Sigma^4K\mathbb{H} \cong K\mathbb{R}$. \square

We will also need to know what the Atiyah–Bott–Shapiro maps do in cohomology. It is quite simple:

Lemma 5.2.4. *The map $\varphi_{\mathbb{F}} : M\text{Spin}_{\mathbb{F}} \rightarrow k\mathbb{F}$ is such that the nonzero class $x \in H^0k\mathbb{F}$ goes to the Thom class $U_{\mathbb{F}} \in H^0M\text{Spin}_{\mathbb{F}}$.*

Proof. Using the fact that all closed zero-manifolds are disjoint unions of points and compact one-manifolds are disjoint unions of line segments and circles, it follows that $\pi_0M\text{Spin}_{\mathbb{F}} \cong \mathbb{Z}$, with a generator being a point with one of two $\text{Spin}_{\mathbb{F}}$ -structures. So if a generator of $\pi_0M\text{Spin}_{\mathbb{F}}$ maps to a generator of $\pi_0k\mathbb{F} \cong \mathbb{Z}$, then we will get an isomorphism in degree zero (integral) homology by the Hurewicz theorem, which then gives an isomorphism in degree zero ($\mathbb{Z}/2\mathbb{Z}$) cohomology.

To see that this statement about π_0 is true, we can take the map $\mathbb{S} \rightarrow M\text{Spin}_{\mathbb{F}}$ classifying such a generator, which is the map $M\xi \rightarrow M\text{Spin}_{\mathbb{F}}$, where ξ is a $\text{Spin}_{\mathbb{F}}(0)$ -vector bundle over a point. We can take the relevant $\text{Cl}_{\mathbb{F}}(0)$ -module μ to be $\text{Cl}_{\mathbb{F}}(0) \cong \mathbb{F}$ (all even-graded). Then as we saw in Proposition 4.2.10, the difference bundle construction gives us $[\mathbb{F}] \in \tilde{K}\mathbb{F}^0(*)$. \square

For $\mathbb{F} = \mathbb{R}, \mathbb{C}$, this can also be shown more formally using the fact that $\varphi_{\mathbb{F}}$ is a map of ring spectra. Since the cohomology of these spectra have coalgebra structures induced by the ring structure, the fact that they are zero in negative degrees and only have one nonzero class in degree zero implies that the counit maps must take this class to 1 and all other homogeneous classes go to zero. Then since the map in cohomology induced by $\varphi_{\mathbb{F}}$ preserves the counit, the result follows.

There is one additional spectrum that we must introduce for the splitting of $M\text{Spin}_{\mathbb{H}}$. This spectrum plays the role of $k\mathbb{H}\langle 8k + 4 \rangle$, but removes a factor of two that appears in the homotopy groups in order to ensure a 2-local equivalence.

Definition 5.2.5. The spectrum F is defined to be the homotopy fiber of the map $k\mathbb{R} \rightarrow H\mathbb{Z}/2\mathbb{Z}$ that classifies the generator of $H^*k\mathbb{R}$.

The homotopy groups and cohomology of this spectrum are readily computable:

Proposition 5.2.6. *The homotopy groups of F are trivial in negative degrees, and if $k \geq 0$, then*

$$\pi_k F \cong \begin{cases} \mathbb{Z} & k = 0, 4 \pmod{8}, \\ \mathbb{Z}/2\mathbb{Z} & k = 1, 2 \pmod{8}, \\ 0 & \text{otherwise.} \end{cases}$$

The cohomology of F is given by

$$H^*F \cong \Sigma^{-1}(\mathcal{A}Sq^1 + \mathcal{A}Sq^2).$$

Proof. The map $k\mathbb{R} \rightarrow H\mathbb{Z}/2\mathbb{Z}$ gives the surjective map $\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$ after applying π_0 using the Hurewicz theorem. Then $\pi_k F$ for all degrees follows immediately from the long exact sequence of the fibration $F \rightarrow ko \rightarrow H\mathbb{Z}/2\mathbb{Z}$. Similarly, we get a long exact sequence in cohomology from this (co)fibration, and the map $k\mathbb{R} \rightarrow H\mathbb{Z}/2\mathbb{Z}$ induces the quotient map $\mathcal{A} \rightarrow \mathcal{A}/(\mathcal{A}Sq^1 + \mathcal{A}Sq^2)$, so H^*F is given by shifting the kernel of this map down by one degree. \square

We denote the generator of H^0F by e_0 and the generator of H^1F by e_1 .

5.3 Construction of the Splitting

So now we have all the pieces of the splitting, and we need to show that they can be arranged together to form the desired result. What this means is building maps from $M\text{Spin}_{\mathbb{F}}$ to our summands, and then showing that we get an isomorphism in cohomology once we add these maps up. The Atiyah–Bott–Shapiro orientation, along with the cohomology of $M\text{Spin}_{\mathbb{F}}$, are the key ingredients that allow us to synthesize all of these maps.

The point of the Atiyah–Bott–Shapiro orientation in this discussion is that, by viewing it as a K-theory class on $M\text{Spin}_{\mathbb{F}}$, we can multiply with it to transfer K-theory classes on $B\text{Spin}_{\mathbb{F}}$ to $M\text{Spin}_{\mathbb{F}}$. As we saw, when $\mathbb{F} = \mathbb{R}, \mathbb{C}$, this is an isomorphism. For $\mathbb{F} = \mathbb{H}$, this is not quite an isomorphism, and it maps real vector bundles to quaternionic ones, but this is sufficient for our purposes. So, let us begin by introducing the K-theory classes on the classifying spaces that we want to transfer over:

Proposition 5.3.1. *Suppose X is a space and $\xi : X \rightarrow BSO$. Then there exist $K\mathbb{R}$ -Pontryagin classes $\pi^i \in K\mathbb{R}^0(X)$, which are characterized uniquely by the following properties:*

1. π^i depends on ξ only up to homotopy.
2. π^i is natural in ξ (it commutes with pulling back $K\mathbb{R}^0$ -classes and maps to BSO).
3. If ξ is induced by a complex line bundle (with its canonical orientation), then

$$\pi^i(\xi) = \begin{cases} 0 & i = 0, i \geq 2, \\ \xi - 2 & i = 1. \end{cases}$$

4. If $\xi_1, \xi_2 : X \rightarrow BSO$, then

$$\sum_{i=0}^{\infty} \pi^i(\xi_1 \oplus \xi_2) t^i = \sum_{i,j=0}^{\infty} \pi^i(\xi_1) \pi^j(\xi_2) t^{i+j}.$$

Proof. See [ABP66]. □

For a multi-index $I = (i_1, \dots, i_n)$, let $\pi^I(\xi) = \pi^{i_1}(\xi) \dots \pi^{i_n}(\xi)$. For such a multi-index, we let $|I| = \sum_{j=1}^n i_j$ be its sum. These classes satisfy additional nice properties in cohomology:

Proposition 5.3.2. *Let I be a multi-index. If $|I|$ is even, then $\pi^I(\xi)$ admits a lift to $k\mathbb{R}\langle 4|I| \rangle$, and if the Pontryagin class $p_I = p_{i_1}(\xi) \dots p_{i_n}(\xi)$ is nonzero in the rational cohomology of X , then the map $X \rightarrow k\mathbb{R}\langle 4|I| \rangle$ can be chosen so that the generator of $H\mathbb{Z}^{4|I|} k\mathbb{R}\langle 4|I| \rangle$ maps to $p_I + \delta \text{Sq}^2 \text{Sq}^1 \alpha_I$, where δ is the Bockstein homomorphism and α_I is a polynomial in the Stiefel-Whitney classes.*

If $|I|$ is odd, then $\pi^I(\xi)$ lifts to $k\mathbb{R}\langle 4|I| - 2 \rangle$. If p_I is not divisible by two in the image of $H\mathbb{Z}^ X \rightarrow H\mathbb{Q}^* X$, then the lift can be taken so that $\text{Sq}^2 x_{4|I|-2} \in H^{4|I|} k\mathbb{R}\langle 4|I| - 2 \rangle$ maps to the reduction of p_I modulo 2 in $H^{4|I|} X$.*

The complexification $\pi^I(\xi)_{\mathbb{C}}$ lifts to $k\mathbb{C}\langle 4|I| \rangle$. Assuming the conditions above are satisfied for ξ , then the lift can be chosen such that the generator of $H\mathbb{Z}^{4|I|} k\mathbb{C}\langle 4|I| \rangle$ maps to p_I .

Proof. See [Sto68, p. 303]. □

So, by letting I range over all partitions, we can generate a bunch of $k\mathbb{R}$ -classes on $B\text{Spin}_{\mathbb{F}}$ by applying π^I to the bundle on $B\text{Spin}_{\mathbb{F}}$ classified by the map $B\text{Spin}_{\mathbb{F}} \rightarrow BSO$. If $\mathbb{F} = \mathbb{C}$, then we complexify them. Then these classes admit lifts to connective covers of $k\mathbb{R}$ and $k\mathbb{C}$ with the cohomology properties described in the proposition above. To see that the requisite properties of the Pontryagin classes p_I are satisfied, one uses the property that all torsion in the integral cohomology of $B\text{Spin}_{\mathbb{F}}$ has order two.

Now that we have the K-theory classes on the classifying spaces $B\text{Spin}_{\mathbb{F}}$, we can use the Atiyah–Bott–Shapiro orientation to transfer them to the spectra $M\text{Spin}_{\mathbb{F}}$. This is done using the **Thom diagonal** map $M\text{Spin}_{\mathbb{F}} \rightarrow (B\text{Spin}_{\mathbb{F}})_{+} \wedge M\text{Spin}_{\mathbb{F}}$. If we have a

then we can form a class $\alpha\varphi_{\mathbb{F}}$ by forming the composition

$$M\mathrm{Spin}_{\mathbb{F}} \longrightarrow (B\mathrm{Spin}_{\mathbb{F}})_+ \wedge M\mathrm{Spin}_{\mathbb{F}} \xrightarrow{\alpha\wedge\varphi_{\mathbb{F}}} k\mathbb{F}_0\langle n \rangle \wedge k\mathbb{F}\langle m \rangle \longrightarrow k\mathbb{F}\langle n+m \rangle$$

Here the leftmost map is the Thom diagonal, and the rightmost map is the map induced by multiplication.

To move on to the proof of the splitting, we need to know what these maps do in cohomology. This can be done by computing the induced map in cohomology for all of the maps above and composing them. It is well-known what the Thom diagonal map does, because it induces the Thom isomorphism. And the map in the middle is easy to understand because we know what the maps α and $\varphi_{\mathbb{F}}$ do in cohomology. So the only unknown is what the multiplication map $k\mathbb{F}_0\langle n \rangle \wedge k\mathbb{F}\langle m \rangle \rightarrow k\mathbb{F}\langle n+m \rangle$ does.

Lemma 5.3.3. *For the maps $k\mathbb{R}\langle 8n \rangle \wedge k\mathbb{R} \rightarrow k\mathbb{R}\langle 8n \rangle$, $k\mathbb{R}\langle 8n+2 \rangle \wedge k\mathbb{R} \rightarrow k\mathbb{R}\langle 8n+2 \rangle$, $k\mathbb{C}\langle n \rangle \wedge k\mathbb{C} \rightarrow k\mathbb{C}\langle n \rangle$, and $k\mathbb{R}\langle 8n \rangle \wedge k\mathbb{H} \rightarrow k\mathbb{H}\langle 8n \rangle$, the nontrivial cohomology class of lowest degree of the codomain maps to the tensor product of the nontrivial cohomology classes of lowest degree for the factors of the smash product in the domain.*

The map $k\mathbb{R}\langle 8n+2 \rangle \wedge k\mathbb{H} \rightarrow k\mathbb{H}\langle 8n+4 \rangle$ factors through the map $\Sigma^{8k+4} \rightarrow k\mathbb{H}\langle 8n+4 \rangle$. The lift $k\mathbb{R}\langle 8n+2 \rangle \wedge k\mathbb{H} \rightarrow \Sigma^{8n+4}$ can be chosen such that $e_0 \mapsto \mathrm{Sq}^2 x_{8k+2} \otimes x_0$ and $e_1 \mapsto x_{8k+2} \otimes \mathrm{Sq}^3 x_0$.

The maps above fall into two distinct classes. The first is where the lowest nontrivial homotopy group of the domain and codomain is isomorphic to \mathbb{Z} . So this class includes $k\mathbb{R}\langle 8n \rangle \wedge k\mathbb{R} \rightarrow k\mathbb{R}\langle 8n \rangle$, $k\mathbb{C}\langle n \rangle \wedge k\mathbb{C} \rightarrow k\mathbb{C}\langle n \rangle$, and $k\mathbb{R}\langle 8n \rangle \wedge k\mathbb{H} \rightarrow k\mathbb{H}\langle 8n \rangle$. The proof is simple in this case, since we can use the known ring/module structure of the homotopy groups of $K\mathbb{R}$ and $K\mathbb{H}$ and the Hurewicz theorem.

The proof for the other class, which includes the maps of the form $k\mathbb{R}\langle 8n+2 \rangle \wedge k\mathbb{R} \rightarrow k\mathbb{R}\langle 8n+2 \rangle$ and $k\mathbb{R}\langle 8n+2 \rangle \wedge k\mathbb{H} \rightarrow k\mathbb{H}\langle 8n+2 \rangle$, involves a few steps. First, one lowers n as much as possible (while keeping it nonnegative) using the fact that $\Sigma^8 K\mathbb{R} \cong K\mathbb{R}$ and $K\mathbb{R} \cong \Sigma^4 K\mathbb{R}$ as $K\mathbb{R}$ -modules. This reduces the problem to only finitely many cases that we need to study. It is important that these are isomorphisms of modules, because this ensures that when we take higher connective covers by suspending, we recover the multiplication maps for the higher connective covers. In the quaternionic case, we can trade $k\mathbb{H}$ with $\Sigma^{-4}k\mathbb{R}\langle 4 \rangle$, so we only need to consider the map $k\mathbb{R}\langle 2 \rangle \wedge k\mathbb{R}\langle 4 \rangle \rightarrow k\mathbb{R}\langle 8 \rangle$.

Then, one computes the action in integral cohomology in low degrees for the zeroth spaces of these spectra. One way to do this is to use the Chern character. Since we have real bundles, we must complexify them to use this. Since the Chern character is multiplicative for tensor products [MS74, Problem 16-B], we can compute all of the low degree Chern classes of the tensor product.

Finally, to see what happens for the spectra, we need to stabilize. By examining the connectivity of these zeroth spaces and using the fact that they are zeroth spaces of Ω -spectra,

we can show that these unstable classes correspond to classes in the integral cohomology of the spectra. For the case $k\mathbb{R}\langle 2 \rangle \wedge k\mathbb{R} \rightarrow k\mathbb{R}\langle 2 \rangle$, we can simply reduce modulo two to get the result. For the case $k\mathbb{R}\langle 2 \rangle \wedge k\mathbb{R}\langle 4 \rangle \rightarrow k\mathbb{R}\langle 8 \rangle$, one shows that the composite of this map with the map $k\mathbb{R}\langle 8 \rangle \rightarrow \Sigma^8 H\mathbb{Z}/2\mathbb{Z}$ classifying the lowest degree nontrivial cohomology class is zero, and thus we get a lift to $\Sigma^8 F$. Then, using some computations in cohomology and obstruction theory, this lift can be chosen to get the desired result.

We summarize the results of this section in the following proposition, which captures all the K-theory classes and cohomological data of this section. Note that for $M\text{Spin}$, we exclude multi-indices containing 1 because $w_2 = 0$ in $B\text{Spin}$, and thus the Pontryagin class p_I vanishes if I contains 1.

Proposition 5.3.4. *If I is a multi-index not containing 1 and $|I|$ is even, then there is a map $M\text{Spin} \rightarrow k\mathbb{R}\langle 4|I| \rangle$ with $x_{4|I|} \mapsto (p_I + \text{Sq}^3 \text{Sq}^1 \alpha_I)U$ for some $\alpha_I \in H^* B\text{Spin}$, and if $|I|$ is odd, then there is a map $M\text{Spin} \rightarrow k\mathbb{R}\langle 4|I| - 2 \rangle$ where $x_{4|I|-2} \mapsto \beta_I U$, and $\text{Sq}^2 \beta_I = p_I$.*

For $M\text{Spin}_{\mathbb{C}}$, for every I , there is a map $M\text{Spin}_{\mathbb{C}} \rightarrow k\mathbb{C}\langle 4|I| \rangle$ such that $x_{4|I|} \mapsto p_I$.

For $M\text{Spin}_{\mathbb{H}}$ and for every I , if $|I|$ is even, then there is a map $M\text{Spin}_{\mathbb{H}} \rightarrow k\mathbb{H}\langle 4|I| \rangle$ with $x_{4|I|} \mapsto (p_I + \text{Sq}^3 \text{Sq}^1 \alpha_I)U$ for some $\alpha_I \in H^ B\text{Spin}_{\mathbb{H}}$, and if $|I|$ is odd, there is a map $M\text{Spin}_{\mathbb{H}} \rightarrow \Sigma^{4|I|} F$ such that $e_0 \mapsto p_I U$ and $e_1 \mapsto \beta_I w_3 U$ for some $\beta_I \in H^* B\text{Spin}_{\mathbb{H}}$.*

5.4 An Isomorphism in Cohomology

Now that we have the maps, we just need to show that they induce an isomorphism in cohomology after we sum them together. This is a daunting cohomology calculation. We will only summarize the conceptually important ideas while minimizing the amount of explicit Steenrod algebra calculations that we do. As we saw above, the structure of the Steenrod modules involved in the splitting really only depends on a finite subalgebra of \mathcal{A} . What makes this calculation reasonable is that, in fact, an even smaller amount of data is enough to show that the splitting is an equivalence.

In the subalgebra \mathcal{A}_1 , the elements $Q_0 := \text{Sq}^1$ and $Q_1 := \text{Sq}^2 \text{Sq}^1 + \text{Sq}^3$ satisfy $Q_i^2 = 0$. So any module over \mathcal{A}_1 becomes can be consider a chain complex with Q_i acting as the differential.

Definition 5.4.1. If M is an \mathcal{A}_1 -module, then $H_k(M; Q_i)$ denotes the homology of M with respect to the differential Q_i . These are referred to as the **Margolis homology groups** of M .

These groups ($\mathbb{Z}/2\mathbb{Z}$ -vector spaces) have the following properties that we will need:

- For any collection of \mathcal{A}_1 -modules M_j , there is a natural isomorphism
$$H_k \left(\bigoplus_j M_j; Q_i \right) \cong \bigoplus_j H_k(M_j; Q_i).$$

- If M_1 and M_2 are \mathcal{A}_1 -modules, then there is a natural isomorphism $H_*(M_1 \otimes M_2; Q_i) \cong H_*(M_1; Q_i) \otimes H_*(M_2; Q_i)$. Here the \mathcal{A}_1 -module structure on $M_1 \otimes M_2$ is given by the Cartan formula, and $H_*(M_1; Q_i) \otimes H_*(M_2; Q_i)$ is understood to be the *graded* tensor product of two graded vector spaces.

While its use in our calculations will be relatively straightforward, these homology groups have a deep theory developed in [Mar83]. Proofs for the above properties can also be found in this book.

So, the strategy to prove the splitting is to sum together all the maps in Proposition 5.3.4 and show that when we take cohomology and then Margolis homology, we get isomorphisms. Then, we carefully pick some cohomology classes such that when sum together the maps of Proposition 5.3.4 and maps $\Sigma^n H\mathbb{Z}/2\mathbb{Z}$ representing these cohomology classes, we get an actual isomorphism in cohomology.

To show that the maps

$$M\text{Spin} \rightarrow \bigvee_{I \in \mathcal{P}_1(2k)} k\mathbb{R}\langle 8k \rangle \vee \bigvee_{I \in \mathcal{P}_1(2k+1)} k\mathbb{R}\langle 8k+2 \rangle, \quad (5.2)$$

$$M\text{Spin}_{\mathbb{C}} \rightarrow \bigvee_{I \in \mathcal{P}} k\mathbb{C}\langle 4|I| \rangle, \quad (5.3)$$

$$M\text{Spin}_{\mathbb{H}} \rightarrow \bigvee_{I \in \mathcal{P}(2k)} k\mathbb{H}\langle 8k \rangle \vee \bigvee_{I \in \mathcal{P}(2k+1)} \Sigma^{8k+4} F \quad (5.4)$$

give isomorphisms in Margolis homology, we compute the dimension of the Margolis homology for each degree of each side (noting that they are equal), and then we show that the maps are surjective. Again, we will only summarize the key points involved in this and not go into detail with these calculations. Let $M_{\mathbb{F}} = H^*M\text{Spin}_{\mathbb{F}}$ and let $\bar{\theta}_{\mathbb{F}}$ be the map of cohomology groups induced by the map of spectra above.

Proposition 5.4.2. *The Margolis homology groups for the Thom spectra $M\text{Spin}_{\mathbb{F}}$ are:*

$$\begin{aligned} H_*(M_{\mathbb{R}}; Q_0) &\cong \mathbb{Z}/2\mathbb{Z}[w_{2^i}^2, f_{2^k} \mid k \geq 2, i \geq 2, i \neq 2^r]U, \\ H_*(M_{\mathbb{R}}; Q_1) &\cong \mathbb{Z}/2\mathbb{Z}[w_{2^i}^2, g_{2^k-2} \mid i \geq 2, i \neq 2^k-1, k \geq 3]U, \\ H_*(M_{\mathbb{C}}; Q_0) &\cong \mathbb{Z}/2\mathbb{Z}[w_{2^i}^2, f_{2^k} \mid i \neq 2^r]U, \\ H_*(M_{\mathbb{C}}; Q_1) &\cong \mathbb{Z}/2\mathbb{Z}[w_{2^i}^2, g_{2^k-2} \mid i \neq 2^k-1, k \geq 2]U, \\ H_*(M_{\mathbb{H}}; Q_0) &\cong \mathbb{Z}/2\mathbb{Z}[w_{2^i}^2, f_{2^k} \mid k \geq 2, i \geq 3, i \neq 2^{r-1} \text{ for } r \geq 2]U, \\ H_*(M_{\mathbb{H}}; Q_1) &\cong \mathbb{Z}/2\mathbb{Z}[w_{2^i}^2, g_{2^k-2} \mid i \neq 2^k-1, k \geq 3]w_2U. \end{aligned}$$

Here the elements f_i and g_i are certain cohomology classes of $B\text{Spin}_{\mathbb{F}}$ with degree i . Their exact value is not relevant outside the calculation above. They are constructed by acting with certain elements of \mathcal{A} on the Thom class U and using the Thom isomorphism to extract

classes on $B\text{Spin}_{\mathbb{F}}$. The point of these classes is that they allow us to break up $M_{\mathbb{F}}$ into a tensor product of polynomial algebras that are closed under the action of Q_i . One then computes the Q_i -homology of these polynomial algebras, which is typically very easy, and then uses the Künneth theorem to get the result. This calculation is carried out in [ABP67] for $\mathbb{F} = \mathbb{R}$, [Sto68, p. 325] for $\mathbb{F} = \mathbb{C}$, and [BM23] for $\mathbb{F} = \mathbb{H}$.

Similarly, we can calculate the Margolis homology groups for the K-theory spectra.

Proposition 5.4.3. *We have the following presentations for the Q_i -homology of various relevant Steenrod modules:*

$$\begin{aligned}
H_*(\mathcal{A}/(\mathcal{A}\text{Sq}^1 + \mathcal{A}\text{Sq}^2); Q_0) &\cong \langle \chi(\text{Sq}^{4k}) \mid k \geq 0 \rangle, \\
H_*(\mathcal{A}/(\mathcal{A}\text{Sq}^1 + \mathcal{A}\text{Sq}^2); Q_1) &\cong \langle \chi(\text{Sq}^{2(\Delta i_1 + \dots + \Delta i_k)}) \mid i_1 > i_2 > \dots > i_k \geq 2 \rangle, \\
H_*(\mathcal{A}/\mathcal{A}\text{Sq}^3; Q_0) &\cong \langle \chi(\text{Sq}^{4k})\text{Sq}^2 \mid k \geq 0 \rangle, \\
H_*(\mathcal{A}/\mathcal{A}\text{Sq}^3; Q_1) &\cong \langle \chi(\text{Sq}^{2(\Delta i_1 + \dots + \Delta i_k)})\text{Sq}^2 \mid i_1 > i_2 > \dots > i_k \geq 2 \rangle, \\
H_*(\mathcal{A}/(\mathcal{A}\text{Sq}^1 + \mathcal{A}\text{Sq}^3); Q_0) &\cong \langle a_{2k} \mid k \geq 0 \rangle, \\
H_*(\mathcal{A}/(\mathcal{A}\text{Sq}^1 + \mathcal{A}\text{Sq}^3); Q_1) &\cong \langle b_k \mid k = (2^{i_1} - 2) + (2^{i_2} - 2) + \dots + (2^{i_k} - 2), \\
&\quad i_1 > i_2 > \dots > i_k \geq 2 \rangle, \\
H_*(\mathcal{A}/(\mathcal{A}\text{Sq}^1 + \mathcal{A}\text{Sq}^5); Q_0) &\cong \langle \chi(\text{Sq}^{4k}) \mid k \geq 0 \rangle, \\
H_*(\mathcal{A}/(\mathcal{A}\text{Sq}^1 + \mathcal{A}\text{Sq}^5); Q_1) &\cong \langle \chi(\text{Sq}^{2(\Delta i_1 + \dots + \Delta i_k)})\text{Sq}^2 \mid i_1 > i_2 > \dots > i_k \geq 2 \rangle, \\
H_*(\mathcal{A}\text{Sq}^1 + \mathcal{A}\text{Sq}^2; Q_0) &\cong \langle \chi(\text{Sq}^{4k})e_0 + \chi(\text{Sq}^{4k-2})\text{Sq}^1e_1 \mid k \geq 0 \rangle, \\
H_*(\mathcal{A}\text{Sq}^1 + \mathcal{A}\text{Sq}^2; Q_1) &\cong \langle \chi(\text{Sq}^{2(\Delta i_1 + \dots + \Delta i_k)})(\text{Sq}^2e_0 + \text{Sq}^1e_1) \\
&\quad \mid i_1 > i_2 > \dots > i_k \geq 2 \rangle.
\end{aligned}$$

To calculate these, one can use the dual Steenrod algebra, which is developed quite thoroughly in [Mil58]. Here χ denotes the antipode of \mathcal{A} , and Δi denotes the sequence $\Delta i = (j_1, j_2, j_3, \dots)$ where $j_k = \delta_{ik}$. The Steenrod squares involving linear combinations of the Δi are understood to be Milnor basis elements for \mathcal{A} (see [Mil58]). The calculations for $\mathcal{A}/(\mathcal{A}\text{Sq}^1 + \mathcal{A}\text{Sq}^2)$, $\mathcal{A}/\mathcal{A}\text{Sq}^3$, and $\mathcal{A}/(\mathcal{A}\text{Sq}^1 + \mathcal{A}\text{Sq}^3)$ are carried out using this method in [ABP67] and [Sto68].

To get the results for $\mathcal{A}/(\mathcal{A}\text{Sq}^1 + \mathcal{A}\text{Sq}^5)$ and $\mathcal{A}\text{Sq}^1 + \mathcal{A}\text{Sq}^2$, there is an exact sequence

$$0 \longrightarrow \Sigma^3 \mathcal{A}/(\mathcal{A}\text{Sq}^1 + \mathcal{A}\text{Sq}^5) \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{A}\text{Sq}^3 \longrightarrow 0$$

where $1 \in \Sigma^3 \mathcal{A}/(\mathcal{A}\text{Sq}^1 + \mathcal{A}\text{Sq}^5)$ maps to $\text{Sq}^3 \in \mathcal{A}$, as well as an exact sequence

$$0 \longrightarrow \mathcal{A}\text{Sq}^1 + \mathcal{A}\text{Sq}^2 \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}/(\mathcal{A}\text{Sq}^1 + \mathcal{A}\text{Sq}^2) \longrightarrow 0$$

Using the explicit description of the connecting homomorphism for the long exact sequence in homology as well as the results we have for the $\mathcal{A}/(\mathcal{A}\text{Sq}^1 + \mathcal{A}\text{Sq}^2)$ and $\mathcal{A}/\mathcal{A}\text{Sq}^3$, one can then compute the homology groups for these modules and the cycles that generate them.

Lemma 5.4.4. *The Q_i -homology groups for the cohomology of each side of Maps 5.2-5.4 have the same dimension in each degree.*

This is straightforward to prove using the combinatorics of partitions and polynomial rings and the Q_i -homology for all the Steenrod modules we calculated above.

Proposition 5.4.5. *Maps $\bar{\theta}_{\mathbb{F}}$ induces isomorphisms in Q_i -homology.*

One way to prove this is to first treat the case $\mathbb{F} = \mathbb{C}$. For this, one can use the fact that $M\text{Spin}_{\mathbb{C}}$ admits the structure of a ring spectrum, and so its cohomology $M_{\mathbb{C}} = H^*M\text{Spin}_{\mathbb{C}}$ has an induced coalgebra structure. The Steenrod module $\mathcal{A}/(\mathcal{A}S\text{q}^1 + \mathcal{A}S\text{q}^3)$ also has the structure of a coalgebra, and the map $\mathcal{A}/(\mathcal{A}S\text{q}^1 + \mathcal{A}S\text{q}^3) \rightarrow M_{\mathbb{C}}$ is a map of coalgebras since the map $M\text{Spin}_{\mathbb{C}} \rightarrow k\mathbb{C}$ is a map of ring spectra. The Q_i -homology groups of these coalgebras have an induced coalgebra structure as well, and so we can show that every generator of $H(M_{\mathbb{C}}; Q_i)$ is in the image of the maps induced by Map 5.3 by applying the comultiplication repeatedly and analyzing the result to see that it must be the case for every generator to be in the image. So we get a surjection, and using the lemma above, we see that the map induces isomorphisms.

Then, for the other two cases, we can use the fact that we have maps $M\text{Spin} \rightarrow M\text{Spin}_{\mathbb{C}} \rightarrow M\text{Spin}_{\mathbb{H}}$ such that in cohomology, Thom classes pull back to Thom classes and Stiefel-Whitney classes pull back to the corresponding Stiefel-Whitney classes. For the real case, there are maps $\mathcal{A}/(\mathcal{A}S\text{q}^1 + \mathcal{A}S\text{q}^3) \rightarrow \mathcal{A}/(\mathcal{A}S\text{q}^1 + \mathcal{A}S\text{q}^2)$ and $\Sigma^2\mathcal{A}/(\mathcal{A}S\text{q}^1 + \mathcal{A}S\text{q}^3) \rightarrow \mathcal{A}/\mathcal{A}S\text{q}^3$. Using these maps, we can form a non-commuting diagram

$$\begin{array}{ccc}
 \bigoplus_{I \in \mathcal{P}} \Sigma^{4|I|} \mathcal{A}/(\mathcal{A}S\text{q}^1 + \mathcal{A}S\text{q}^3) & \longrightarrow & M_{\mathbb{C}} \\
 \downarrow & & \downarrow \\
 \bigoplus_{I \in \mathcal{P}_1(2k)} \Sigma^{8k} \mathcal{A}/(\mathcal{A}S\text{q}^1 + \mathcal{A}S\text{q}^2) \oplus \bigoplus_{I \in \mathcal{P}_1(2k+1)} \Sigma^{8k+2} \mathcal{A}/\mathcal{A}S\text{q}^3 & \longrightarrow & M_{\mathbb{R}}
 \end{array}$$

However, when we apply the functors $H_*(-; Q_i)$, the resulting diagrams do commute. This can be shown by checking by hand the image of each generator of $\bigoplus_{I \in \mathcal{P}} \Sigma^{4|I|} \mathcal{A}/(\mathcal{A}S\text{q}^1 + \mathcal{A}S\text{q}^2)$ along both paths. The top map goes to isomorphisms, and the right map goes to surjections, so the bottom map must become a surjection as well. Then, since the dimensions in each degree are equal, the bottom map induces isomorphisms.

For the quaternionic case, there are maps $\mathcal{A}/(\mathcal{A}S\text{q}^1 + \text{Sq}^5) \rightarrow \mathcal{A}/(\mathcal{A}S\text{q}^1 + \text{Sq}^3)$ and $\mathcal{A}S\text{q}^1 + \mathcal{A}S\text{q}^2 \rightarrow \mathcal{A}/(\mathcal{A}S\text{q}^1 + \mathcal{A}S\text{q}^3)$. Using some basic Q_i -homology calculations, it can be

shown that these induce injections for Q_i -homology. Then the diagram

$$\begin{array}{ccc}
\bigoplus_{I \in \mathcal{P}(2k)} \Sigma^{8k} \mathcal{A}/(\mathcal{A}S\mathfrak{q}^1 + \mathcal{A}S\mathfrak{q}^5) \oplus \bigoplus_{I \in \mathcal{P}(2k+1)} \Sigma^{8k+4}(\mathcal{A}S\mathfrak{q}^1 + \mathcal{A}S\mathfrak{q}^2) & \longrightarrow & M_{\mathbb{H}} \\
\downarrow & & \downarrow \\
\bigoplus_{I \in \mathcal{P}} \Sigma^{4|I|} \mathcal{A}/(\mathcal{A}S\mathfrak{q}^1 + \mathcal{A}S\mathfrak{q}^3) & \longrightarrow & M_{\mathbb{C}}
\end{array}$$

does not commute, but the diagrams we get by applying the functors $H_*(-; Q_i)$ do. The leftmost arrow induces injections, and the bottom arrow goes to isomorphisms, so the top arrow must be injective after we apply $H_*(-; Q_i)$. And then by the dimension argument, we see that the top arrow must induce isomorphisms.

Now that we know the maps induce isomorphisms of Q_i -homology, we want to locate some ordinary cohomology classes such that adding in the maps $M\text{Spin}_{\mathbb{F}} \rightarrow \Sigma^n H\mathbb{Z}/2\mathbb{Z}$ gives an actual isomorphism in cohomology. First, we show how these classes are chosen.

Let $\mathcal{A}_+ \subset \mathcal{A}$ be the submodule generated by the elements of positive degree. Then $\mathcal{A}_+ M_{\mathbb{F}} \subset M_{\mathbb{F}}$ is a submodule, and so we can form the composite map $q_{\mathbb{F}} \circ \bar{\theta}_{\mathbb{F}}$, where $q_{\mathbb{F}} : M_{\mathbb{F}} \rightarrow M_{\mathbb{F}}/\mathcal{A}_+ M_{\mathbb{F}}$ is the quotient map. Let $c_{\mathbb{F}} : M_{\mathbb{F}}/\mathcal{A}_+ M_{\mathbb{F}} \rightarrow R_{\mathbb{F}}$ be the cokernel of this composite. Choose a set $Z_{\mathbb{F}} \subset M_{\mathbb{F}}$ of homogeneous elements such that $(c_{\mathbb{F}} \circ q_{\mathbb{F}})(Z_{\mathbb{F}})$ is a basis for $R_{\mathbb{F}}$. These cohomology classes are classified by maps $M\text{Spin}_{\mathbb{F}} \rightarrow \Sigma^n H\mathbb{Z}/2\mathbb{Z}$, so we can form maps

$$\begin{aligned}
A_{\mathbb{R}} : M\text{Spin} &\rightarrow \bigvee_{I \in \mathcal{P}_1(2k)} k\mathbb{R}\langle 8k \rangle \vee \bigvee_{I \in \mathcal{P}_1(2k+1)} k\mathbb{R}\langle 8k+2 \rangle \vee \bigvee_{z \in Z_{\mathbb{R}}} \Sigma^{|z|} H\mathbb{Z}/2\mathbb{Z}, \\
A_{\mathbb{C}} : M\text{Spin}_{\mathbb{C}} &\rightarrow \bigvee_{I \in \mathcal{P}} k\mathbb{C}\langle 4|I| \rangle \vee \bigvee_{z \in Z_{\mathbb{C}}} \Sigma^{|z|} H\mathbb{Z}/2\mathbb{Z} \\
A_{\mathbb{H}} : M\text{Spin}_{\mathbb{H}} &\rightarrow \bigvee_{I \in \mathcal{P}(2k)} k\mathbb{H}\langle 8k \rangle \vee \bigvee_{I \in \mathcal{P}(2k+1)} \Sigma^{8k+4} F \vee \bigvee_{z \in Z_{\mathbb{H}}} \Sigma^{|z|} H\mathbb{Z}/2\mathbb{Z}
\end{aligned}$$

Let $\theta_{\mathbb{F}}$ be the map in cohomology induced by these maps. We want to show that $\theta_{\mathbb{F}}$ is an isomorphism. It is immediate from the definition of $Z_{\mathbb{F}}$ that $\theta_{\mathbb{F}}$ is surjective, so we only need to show that it is injective.

Let $N_{\mathbb{F}}$ denote the domain of $\theta_{\mathbb{F}}$. The idea is to filter $N_{\mathbb{F}}$ and $M_{\mathbb{F}}$ in such a way that we can prove show that we have an isomorphism for each step in the filtration. By construction, $N_{\mathbb{F}}$ is a direct sum of many Steenrod modules (i.e. the cohomology of each wedge summand in the codomain of $A_{\mathbb{F}}$). Let $N_{\mathbb{F}}^{[n]} \subset N_{\mathbb{F}}$ be the direct sum of all summands that are nonzero in degrees less than or equal to n , and let $M_{\mathbb{F}}^{[n]} \subseteq M_{\mathbb{F}}$ be $\theta_{\mathbb{F}}(N_{\mathbb{F}}^{[n]})$. Then $\theta_{\mathbb{F}}$ induces a family of maps $\lambda_{\mathbb{F}}^n : N_{\mathbb{F}}/N_{\mathbb{F}}^{[n-1]} \rightarrow M_{\mathbb{F}}/M_{\mathbb{F}}^{[n-1]}$. The module $N_{\mathbb{F}}/N_{\mathbb{F}}^{[n-1]}$ is canonically isomorphic to the sum of all the summands of $N_{\mathbb{F}}$ that are trivial in degrees less than n , and $N_{\mathbb{F}}^{[n]}/N_{\mathbb{F}}^{[n-1]}$

is isomorphic to the sum of all the summands trivial in degrees less than n and nontrivial in degree n . Now, every summand is isomorphic to $M_{\mathcal{A}} := \mathcal{A} \otimes_{\mathcal{A}_1} M$ for some \mathcal{A}_1 -module M , where $\mathcal{A}_1 \subset \mathcal{A}$ is the sub-Hopf algebra generated by Sq^1 and Sq^2 . We define $P_{\mathbb{F},n} \subset N_{\mathbb{F}}^{[n]}/N_{\mathbb{F}}^{[n-1]}$ to be the sub \mathcal{A}_1 -module $\bigoplus X$, where X ranges over the \mathcal{A}_1 -modules such that $M = X_{\mathcal{A}}$, M being one of our summands decomposing $N_{\mathbb{F}}^{[n]}/N_{\mathbb{F}}^{[n-1]}$.

A routine application of the five lemma and the long exact sequence in homology implies that if $\theta_{\mathbb{F}}$ restricts to an isomorphism $N_{\mathbb{F}}^{[n-1]} \rightarrow M_{\mathbb{F}}^{[n-1]}$, then $\lambda_{\mathbb{F}}^n$ induces isomorphisms in Q_i -homology. This is where we use that $\bar{\theta}$ induces isomorphisms in Q_i -homology. Since $H_*(\mathcal{A}; Q_i) \cong 0$, we have that θ induces isomorphisms in Q_i -homology as well. This then implies that the restriction of $\lambda_{\mathbb{F}}^n$ to $P_{\mathbb{F},n}$ is injective. One proves this by considering the possible elements of $P_{\mathbb{F},n}$ in terms of direct sums of \mathcal{A}_1 -modules, and showing that if the image of this were zero, then we could find a cycle in $H_*(M_{\mathbb{F}}/M_{\mathbb{F}}^{[n-1]}; Q_i)$ that is not in the image the map induced by $\lambda_{\mathbb{F}}^n$, a contradiction.

From this, it is possible to show that $\lambda_{\mathbb{F}}^n$ is injective when restricted to $N_{\mathbb{F}}^{[n]}/N_{\mathbb{F}}^{[n-1]}$. Since $N_{\mathbb{F}}^{[n]}/N_{\mathbb{F}}^{[n-1]}$ is generated as an \mathcal{A} -module by $P_{\mathbb{F},n}$, we can write a general element of $N_{\mathbb{F}}^{[n]}/N_{\mathbb{F}}^{[n-1]}$ in terms of elements of $P_{\mathbb{F},n}$. Then, by carefully rewriting this element in a certain form and using the coalgebra (or comodule over $M_{\mathbb{R}}$) structure of $M_{\mathbb{F}}$, one can show its image cannot be zero. So $\lambda_{\mathbb{F}}^n$ is injective when restricted to $N_{\mathbb{F}}^{[n]}/N_{\mathbb{F}}^{[n-1]}$.

Finally, by applying the four lemma to the diagrams

$$\begin{array}{ccccccc}
0 & \longrightarrow & N_{\mathbb{F}}^{[n-1]} & \longrightarrow & N_{\mathbb{F}}^{[n]} & \longrightarrow & N_{\mathbb{F}}^{[n]}/N_{\mathbb{F}}^{[n-1]} \\
\downarrow & & \downarrow \theta_{\mathbb{F}} & & \downarrow \theta_{\mathbb{F}} & & \downarrow \lambda_{\mathbb{F}}^n \\
0 & \longrightarrow & M_{\mathbb{F}}^{[n-1]} & \longrightarrow & M_{\mathbb{F}}^{[n]} & \longrightarrow & M_{\mathbb{F}}^{[n]}/M_{\mathbb{F}}^{[n-1]}
\end{array}$$

we see that the map $\theta_{\mathbb{F}}$ is injective when restricted to $N_{\mathbb{F}}^{[n]}$, provided it restricts to an isomorphism from $N_{\mathbb{F}}^{[n-1]}$ to $M_{\mathbb{F}}^{[n-1]}$. But since $\theta_{\mathbb{F}}$ is surjective, this implies it gives an isomorphism $N_{\mathbb{F}}^{[n]} \rightarrow M_{\mathbb{F}}^{[n]}$. By induction on n , it follows that $\theta_{\mathbb{F}}$ restricts to isomorphisms $N_{\mathbb{F}}^{[n]} \rightarrow M_{\mathbb{F}}^{[n]}$ for all n . And since $N_{\mathbb{F}}$ and $M_{\mathbb{F}}$ are the union of $N_{\mathbb{F}}^{[n]}$ and $M_{\mathbb{F}}^{[n]}$, respectively, it follows that $\theta_{\mathbb{F}}$ is an isomorphism.

Putting everything together, we get the following theorem:

Theorem 5.4.6. *There are sets of homogeneous elements $Z_{\mathbb{F}} \subset H^* M\text{Spin}_{\mathbb{F}}$ such that the maps*

$$A_{\mathbb{R}} : M\text{Spin} \rightarrow \bigvee_{I \in \mathcal{P}_1(2k)} k\mathbb{R}\langle 8k \rangle \vee \bigvee_{I \in \mathcal{P}_1(2k+1)} k\mathbb{R}\langle 8k+2 \rangle \vee \bigvee_{z \in Z_{\mathbb{R}}} \Sigma^{|z|} H\mathbb{Z}/2\mathbb{Z},$$

$$\begin{aligned}
A_{\mathbb{C}} : M\mathrm{Spin}_{\mathbb{C}} &\rightarrow \bigvee_{I \in \mathcal{P}} k\mathbb{C}\langle 4|I| \rangle \vee \bigvee_{z \in Z_{\mathbb{C}}} \Sigma^{|z|} H\mathbb{Z}/2\mathbb{Z} \\
A_{\mathbb{H}} : M\mathrm{Spin}_{\mathbb{H}} &\rightarrow \bigvee_{I \in \mathcal{P}(2k)} k\mathbb{H}\langle 8k \rangle \vee \bigvee_{I \in \mathcal{P}(2k+1)} \Sigma^{8k+4} F \vee \bigvee_{z \in Z_{\mathbb{H}}} \Sigma^{|z|} H\mathbb{Z}/2\mathbb{Z}
\end{aligned}$$

are 2-local equivalences.

Note 5.4.7. Technically we have only shown enough to prove that these maps induce 2-complete equivalences. To deduce 2-local equivalences, it is necessary to show that the homotopy groups of all these Thom spectra are finitely generated in each degree. However, we show this in the following section where we discuss the bordism groups.

5.5 Determination of Bordism Groups

With the splitting in place, one can completely determine the bordism groups associated to $\mathrm{Spin}^{\mathbb{F}}$. This is due to the following fact which implies the 2-local homotopy groups are enough to piece together the ordinary homotopy groups:

Proposition 5.5.1. *The homotopy groups of $M\mathrm{Spin}_{\mathbb{F}}$ are finitely generated and do not contain torsion of odd order.*

Proof. The homotopy groups are finitely generated because $\mathrm{Spin}_{\mathbb{F}}$ is a compact Lie group. For odd torsion, recall the fiber sequences $B\mathbb{Z}/2\mathbb{Z} \rightarrow B\mathrm{Spin} \times BU_{\mathbb{F}} \rightarrow B\mathrm{Spin}_{\mathbb{F}}$. When we invert two, then these become equivalences $BSO \times BU_{\mathbb{F}} \rightarrow B\mathrm{Spin}_{\mathbb{F}}$. These induce equivalences $M\mathrm{SO} \wedge (BU_{\mathbb{F}})_{+} \rightarrow M\mathrm{Spin}_{\mathbb{F}}$ after inverting two. So the homotopy groups of $M\mathrm{Spin}_{\mathbb{F}}$ and $M\mathrm{SO}_{*}BU_{\mathbb{F}}$ are isomorphic after inverting two. In the case $\mathbb{F} = \mathbb{R}$, $BU_{\mathbb{F}} \cong *$ when two is inverted, and since Ω_{*}^{SO} has no odd torsion (see [Sto68, p. 180]), neither does $\pi_{*}M\mathrm{Spin}$. For $\mathbb{F} = \mathbb{C}, \mathbb{H}$, one can use the Atiyah-Hirzebruch spectral sequence to show that there is no torsion. For an example of this method (applied to $\mathrm{Spin} \times_{\mathbb{Z}/2\mathbb{Z}} \mathrm{U}(2)$), see [Deb21]. \square

We know how the K-theory summands contribute to the homotopy groups of $M\mathrm{Spin}^{\mathbb{F}}$ and can easily determine how many there are, but we don't have an explicit determination of where the $H\mathbb{Z}/2\mathbb{Z}$ and $H\mathbb{Z}$ summands occur. However, this is quite easy to do computationally.

Definition 5.5.2. If M is a Steenrod module, then the **Poincaré polynomial** $P(M)$ of M is the polynomial whose coefficient in degree k is the dimension of M^k .

By taking Poincaré polynomials of each Steenrod module in the isomorphisms given by the Anderson–Brown–Peterson splitting, we get an equation:

$$P(H^{*}M\mathrm{Spin}^{\mathbb{F}}) = P(\mathcal{A})X + P(\mathcal{A}/\mathcal{A}\mathrm{Sq}^1)Y + Z.$$

Here Z is the Poincaré polynomial of the K-theory summands, X is the polynomial whose degree k coefficient is the number of $\Sigma^k H\mathbb{Z}/2\mathbb{Z}$ summands of $M\text{Spin}^{\mathbb{F}}$, and Y is the polynomial whose degree k coefficient is the number of $\Sigma^k H\mathbb{Z}$ summands of $M\text{Spin}^{\mathbb{F}}$. We know from above that either X or Y is zero, so we can solve for the nonzero polynomial using polynomial division. By working out the details of this, one can compute the location of the Eilenberg-MacLane summands to achieve a complete description of the $\text{Spin}^{\mathbb{F}}$ bordism groups that can be computed algorithmically. A computer program implementing this computation is available at [Buc23]. The bordism groups, in low degrees, are shown in the tables below:

n	$\pi_n M\text{Spin}$	n	$\pi_n M\text{Spin}_{\mathbb{C}}$	n	$\pi_n M\text{Spin}_{\mathbb{H}}$
0	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}
1	$\mathbb{Z}/2\mathbb{Z}$	1	0	1	0
2	$\mathbb{Z}/2\mathbb{Z}$	2	\mathbb{Z}	2	0
3	0	3	0	3	0
4	\mathbb{Z}	4	\mathbb{Z}^2	4	\mathbb{Z}^2
5	0	5	0	5	$\mathbb{Z}/2\mathbb{Z}^2$
6	0	6	\mathbb{Z}^2	6	$\mathbb{Z}/2\mathbb{Z}^2$
7	0	7	0	7	0
8	\mathbb{Z}^2	8	\mathbb{Z}^4	8	\mathbb{Z}^4
9	$\mathbb{Z}/2\mathbb{Z}^2$	9	0	9	$\mathbb{Z}/2\mathbb{Z}$
10	$\mathbb{Z}/2\mathbb{Z}^3$	10	$\mathbb{Z}^4 \oplus \mathbb{Z}/2\mathbb{Z}$	10	$\mathbb{Z}/2\mathbb{Z}$

Table 5.1: Homotopy groups of the Thom spectra associated to $\text{Spin}_{\mathbb{F}}$, which are isomorphic to the $\text{Spin}_{\mathbb{H}}$ bordism groups.

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